Granger Causality: Basic Theory and Application to Neuroscience

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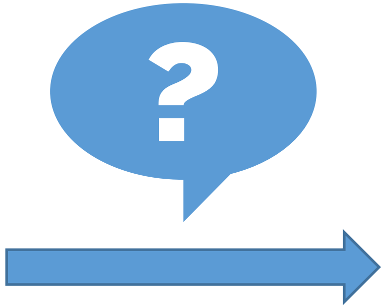
Contents

1. Bivariate Granger Causality

Employment Rate



Stock Price

1.1 Autoregressive Representation

Given two stochastic processes X_t and Y_t, assume they are jointly stationary.

$$X_t = \sum_{j=1}^{\infty} a_{1j} X_{t-j} + \epsilon_{1t}, \quad \mathrm{var}(\epsilon_{1t}) = \Sigma_1. \tag{1}$$

$$Y_t = \sum_{j=1}^{\infty} d_{1j} Y_{t-j} + \eta_{1t}, \quad \mathrm{var}(\eta_{1t}) = \Gamma_1. \tag{2}$$

$$X_t = \sum_{j=1}^{\infty} a_{2j} X_{t-j} + \sum_{j=1}^{\infty} b_{2j} Y_{t-j} + \epsilon_{2t}, \quad \mathrm{var}(\varepsilon_{2t}) = \boldsymbol{\Sigma}_2 \tag{3}$$

$$Y_t = \sum_{j=1}^{\infty} c_{2j} X_{t-j} + \sum_{j=1}^{\infty} d_{2j} Y_{t-j} + \eta_{2t}, \quad \mathrm{var}(\eta_{2t}) = \Gamma_2 \tag{4}$$

Jointly stationary properties:

1. $E(X_t)$ **and** $\mathrm{E}(Y_t)$ **does not depend on t**
2. $\forall\, t, s, \exists\, cov(x_t, x_{t+s})$ **depends on s but not on t**

Where noise terms are uncorrelated over time and their contemporaneous covariance matrix is

$$\Sigma = \begin{pmatrix} \Sigma_2 & \Upsilon_2 \\ \Upsilon_2 & \Gamma_2 \end{pmatrix}$$

, and $\gamma_2 = \mathrm{cov}(\varepsilon_{2t}, \eta_{2t})$

Granger's Definition on Causal Influence:

If Σ_2 is less than Σ_1 in some suitable statistical sense, then Y_t is said to have a casual influence on X_t

Unidirectional Causality from Y_t and X_t: $F_{Y\to X} = \ln\frac{\Sigma_1}{\Sigma_2}$

Intuitions behind the definition of causality

1. When $|\Sigma_1| \geq |\Sigma_2|$, this measure is non-negative
2. This measure is invariant w.r.t scaling of X and Y
3. If $F_{Y\to X} = 0$, $\Sigma_1 = \Sigma_2$, implies $b_{2j} = 0$, that states Y does not cause X.

$$X_t = \sum_{j=1}^{\infty} a_{1j} X_{t-j} + \epsilon_{1t}, \quad \mathrm{var}(\epsilon_{1t}) = \Sigma_1. \quad (1)$$

$$Y_t = \sum_{j=1}^{\infty} d_{1j} Y_{t-j} + \eta_{1t}, \quad \mathrm{var}(\eta_{1t}) = \Gamma_1. \quad (2)$$

$$X_t = \sum_{j=1}^{\infty} a_{2j} X_{t-j} + \sum_{j=1}^{\infty} b_{2j} Y_{t-j} + \epsilon_{2t}, \quad \mathrm{var}(\varepsilon_{2t}) = \Sigma_2 \quad (3)$$

$$Y_t = \sum_{j=1}^{\infty} c_{2j} X_{t-j} + \sum_{j=1}^{\infty} d_{2j} Y_{t-j} + \eta_{2t}, \quad \mathrm{var}(\eta_{2t}) = \Gamma_2 \quad (4)$$

$$\Sigma = \begin{pmatrix} \Sigma_2 & \Upsilon_2 \\ \Upsilon_2 & \Gamma_2 \end{pmatrix}, \text{ and } \gamma_2 = \mathrm{cov}(\varepsilon_{2t}, \eta_{2t})$$

1.2 Time Domain Causality

Interdependence Causality

$$F_{X,Y} = ln\frac{\Sigma_1 \Gamma_1}{|\Sigma|}$$

$F_{X,Y} = 0$ when X and Y are independent

$F_{X,Y} > 0$ when X and Y are not independent

Unidirectional Causality

$$\mathbf{F}_{Y\to X} = \ln\frac{\Sigma_1}{\Sigma_2} \ \& \ \mathbf{F}_{X\to Y} = \ln\frac{\Gamma_1}{\Gamma_2}$$

$F_{Y\to X} = 0$ when there is no causal influence from Y to X

$F_{Y\to X} > 0$ when there is causal influence from Y to X

Instantaneous Causality

$$F_{X\cdot Y} = ln\frac{\Sigma_2 \Gamma_2}{|\Sigma|}$$

$F_{Y\cdot X} = 0$ when there is no instantaneous causality from Y to X

$F_{Y\cdot X} > 0$ when there is instantaneous from Y to X

$$F_{X,Y} = \mathbf{F}_{Y\to X} + \mathbf{F}_{X\to Y} + F_{X\cdot Y}$$

Interdependence Causality — **Unidirectional Causality** — **Instantaneous Causality**

$$X_t = \sum_{j=1}^{\infty} a_{1j} X_{t-j} + \epsilon_{1t}, \quad \mathrm{var}(\epsilon_{1t}) = \Sigma_1. \tag{1}$$

$$Y_t = \sum_{j=1}^{\infty} d_{1j} Y_{t-j} + \eta_{1t}, \quad \mathrm{var}(\eta_{1t}) = \Gamma_1. \tag{2}$$

$$X_t = \sum_{j=1}^{\infty} a_{2j} X_{t-j} + \sum_{j=1}^{\infty} b_{2j} Y_{t-j} + \epsilon_{2t}, \ \mathrm{var}(\varepsilon_{2t}) = \Sigma_2 \tag{3}$$

$$Y_t = \sum_{j=1}^{\infty} c_{2j} X_{t-j} + \sum_{j=1}^{\infty} d_{2j} Y_{t-j} + \eta_{2t}, \ \mathrm{var}(\eta_{2t}) = \Gamma_2 \tag{4}$$

$$\Sigma = \begin{pmatrix} \Sigma_2 & \Upsilon_2 \\ \Upsilon_2 & \Gamma_2 \end{pmatrix}, \text{ and } \gamma_2 = \mathrm{cov}(\varepsilon_{2t}, \eta_{2t})$$

1.3 Example

Consider the following AR(2) model:

$X_t = a_1 * X_{t-1} - \mathrm{a}_2 * X_{t-2} + \varepsilon_t$

$X_t = a_3 * X_{t-1} - \mathrm{a}_4 * X_{t-2} + b_1 * Y_{t-1} - \mathrm{b}_2 * Y_{t-2} + \eta_t$

If $\varepsilon_t \sim N(0,1)$, $\eta_t \sim N(0, 0.7)$, does X_t has a causal influence on Y_t?

1.3 Example

Consider the following AR(2) model:

$X_t = a_1 * X_{t-1} - \mathrm{a}_2 * X_{t-2} + \varepsilon_t$

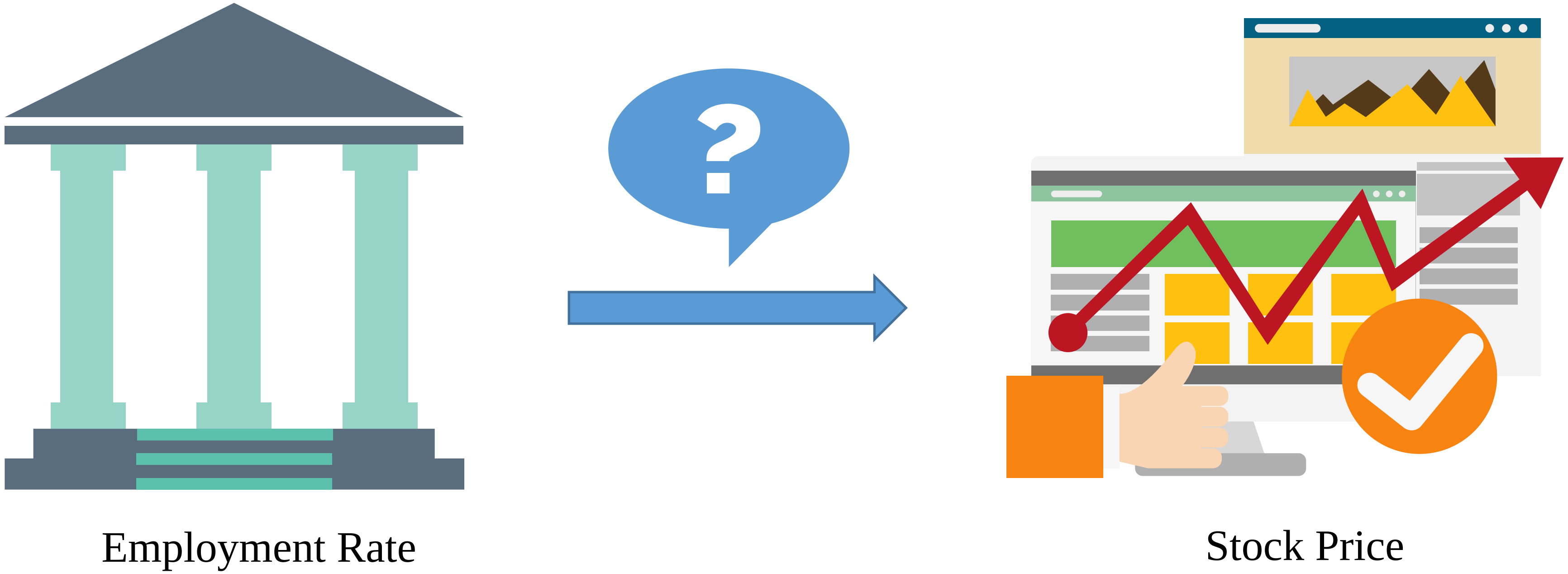
$X_t = a_3 * X_{t-1} - \mathrm{a}_4 * X_{t-2} + b_1 * Y_{t-1} - \mathrm{b}_2 * Y_{t-2} + \eta_t$

If $\varepsilon_t \sim N(0,1)$, $\eta_t \sim N(0, 0.7)$, does X_t has a causal influence on Y_t?

X_t has a causal influence on Y_t.

Since $\mathrm{F}_{X \to Y} = \ln \frac{var(\varepsilon_t)}{Var(\eta_t)} = ln \frac{1}{0.7} > \mathbf{0}$

1.4 Frequency Domain Formulation



Does the employment rate causes stock price weekly/monthly/annually?

1.4 Frequency Domain Formulation

- Three properties on frequency domain formulation:

1. $f_{Y\to X}(\omega) \geq 0$
2. $\frac{1}{2\pi}\int_{-\pi}^{\pi} f_{Y\to X}(\omega)\mathrm{d}\omega = F_{Y\to X}$
3. $F_{Y\to X} = 0 \Leftrightarrow f_{Y\to X}(\omega) = 0 \ \forall\omega.$

1.5 Fourier Transforming on the Lag Operator

$$X_t = \sum_{j=1}^{\infty} a_{2j} X_{t-j} + \sum_{j=1}^{\infty} b_{2j} Y_{t-j} + \epsilon_{2t},$$

$$Y_t = \sum_{j=1}^{\infty} c_{2j} X_{t-j} + \sum_{j=1}^{\infty} d_{2j} Y_{t-j} + \eta_{2t},$$

Rewrite AR expression in terms of lag operator L, where L satisfies $LX_t = X_{t-1}$

$$\begin{pmatrix} a_2(L) & b_2(L) \\ c_2(L) & d_2(L) \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \epsilon_{2t} \\ \eta_{2t} \end{pmatrix}$$

Perform Fourier transforming on both sides

Coefficient Matrix A(ω)

$$\begin{pmatrix} a_2(\omega) & b_2(\omega) \\ c_2(\omega) & d_2(\omega) \end{pmatrix} \begin{pmatrix} X(\omega) \\ Y(\omega) \end{pmatrix} = \begin{pmatrix} E_x(\omega) \\ E_y(\omega) \end{pmatrix}$$

$$a_2(\omega) = 1 - \sum_{j=1}^{\infty} a_{2j} e^{-i\omega j}, \quad b_2(\omega) = -\sum_{j=1}^{\infty} b_{2j} e^{-i\omega j},$$

$$c_2(\omega) = -\sum_{j=1}^{\infty} c_{2j} e^{-i\omega j}, \quad d_2(\omega) = 1 - \sum_{j=1}^{\infty} d_{2j} e^{-i\omega j}.$$

Recasting

Transfer Matrix H(ω) = $A^{-1}(\omega)$

$$\begin{pmatrix} X(\omega) \\ Y(\omega) \end{pmatrix} = \begin{pmatrix} H_{xx}(\omega) & H_{xy}(\omega) \\ H_{yx}(\omega) & H_{yy}(\omega) \end{pmatrix} \begin{pmatrix} E_x(\omega) \\ E_y(\omega) \end{pmatrix}$$

$$H_{xx}(\omega) = \frac{1}{\det \mathbf{A}} d_2(\omega), \quad H_{xy}(\omega) = -\frac{1}{\det \mathbf{A}} b_2(\omega),$$

$$H_{yx}(\omega) = -\frac{1}{\det \mathbf{A}} c_2(\omega), \quad H_{yy}(\omega) = \frac{1}{\det \mathbf{A}} a_2(\omega).$$

1.6 Interdependence Spectral Domain Causality

$$\begin{pmatrix} X(\omega) \\ Y(\omega) \end{pmatrix} = \begin{pmatrix} H_{xx}(\omega) & H_{xy}(\omega) \\ H_{yx}(\omega) & H_{yy}(\omega) \end{pmatrix} \begin{pmatrix} E_x(\omega) \\ E_y(\omega) \end{pmatrix}$$

$$H_{xx}(\omega) = \frac{1}{\det \mathbf{A}} d_2(\omega), \quad H_{xy}(\omega) = -\frac{1}{\det \mathbf{A}} b_2(\omega),$$
$$H_{yx}(\omega) = -\frac{1}{\det \mathbf{A}} c_2(\omega), \quad H_{yy}(\omega) = \frac{1}{\det \mathbf{A}} a_2(\omega).$$

Define spectral matrix for transfer function:

$$S(\omega) = H(\omega)\Sigma H^*(\omega) = \begin{pmatrix} S_{XX}(\omega) & S_{XY}(\omega) \\ S_{YX}(\omega) & S_{YY}(\omega) \end{pmatrix}$$

where * denotes complex conjugate and matrix transpose

$$X_t = \sum_{j=1}^{\infty} a_{2j} X_{t-j} + \sum_{j=1}^{\infty} b_{2j} Y_{t-j} + \epsilon_{2t}, \quad \mathrm{var}(\varepsilon_{2t}) = \mathbf{\Sigma}_2$$

$$Y_t = \sum_{j=1}^{\infty} c_{2j} X_{t-j} + \sum_{j=1}^{\infty} d_{2j} Y_{t-j} + \eta_{2t}, \quad \mathrm{var}(\eta_{2t}) = \Gamma_2$$

$$\mathbf{\Sigma} = \begin{pmatrix} \Sigma_2 & \Upsilon_2 \\ \Upsilon_2 & \Gamma_2 \end{pmatrix}, \text{ and } \gamma_2 = \mathrm{cov}(\varepsilon_{2t}, \eta_{2t})$$

Interdependence Causality:

$$f_{X,Y}(\omega) = \ln \frac{S_{xx}(\omega) S_{yy}(\omega)}{|\mathbf{S}(\omega)|}$$

If X_t and Y_t are independent, $S_{XY}(\omega)$ and $S_{YX}(\omega)$ are zero;

$$f_{X,Y}(\omega) = 0$$

1.7 Unidirectional Frequency Domain Causality

Total power of X_t time series

$$S_{xx}(\omega) = H_{xx}(\omega)\Sigma_2 H^*_{xx}(\omega) + 2\Upsilon_2 \mathrm{Re}(H_{xx}(\omega)H^*_{xy}(\omega)) + H_{xy}(\omega)\Gamma_2 H^*_{xy}(\omega)$$

"Intrinsic Term": Noise Term drives the X_t time series

"Causal Term": Noise Term drives the Y_t time series

γ_2 = 0 means there is no instantaneous causality between X_t and Y_t

How to remove the cross term when $\boldsymbol{\gamma_2}$ is not zero?

1.8 Spectral Normalization: Y→ X Case

TBSI 清华-伯克利深圳学院 Tsinghua-Berkeley Shenzhen Institute

Result after Fourier Transforming

$$\begin{pmatrix} a_2(\omega) & b_2(\omega) \\ c_2(\omega) & d_2(\omega) \end{pmatrix}\begin{pmatrix} X(\omega) \\ Y(\omega) \end{pmatrix} = \begin{pmatrix} E_x(\omega) \\ E_y(\omega) \end{pmatrix}$$

Left–multiplying the normalization matrix P

$$\begin{pmatrix} 1 & 0 \\ -\frac{\gamma_2}{\Sigma_2} & 1 \end{pmatrix}$$

$$\begin{pmatrix} a_2(\omega) & b_2(\omega) \\ c_3(\omega) & d_3(\omega) \end{pmatrix}\begin{pmatrix} X(\omega) \\ Y(\omega) \end{pmatrix} = \begin{pmatrix} E_x(\omega) \\ \tilde{E}_y(\omega) \end{pmatrix}$$

where $c_3(\omega) = c_2(\omega) - \frac{\Upsilon_2}{\Sigma_2} a_2(\omega)$

$$d_3(\omega) = d_2(\omega) - \frac{\Upsilon_2}{\Sigma_2} b_2(\omega).$$

$$\tilde{E}_y(\omega) = E_y(\omega) - \frac{\gamma_2}{\Sigma_2} E_x(\omega)$$

$$\tilde{\mathbf{H}}(\omega) = \begin{pmatrix} \tilde{H}_{xx}(\omega) & \tilde{H}_{xy}(\omega) \\ \tilde{H}_{yx}(\omega) & \tilde{H}_{yy}(\omega) \end{pmatrix} = \frac{1}{\det \tilde{\mathbf{A}}} \begin{pmatrix} d_3(\omega) & -b_2(\omega) \\ -c_3(\omega) & a_2(\omega) \end{pmatrix}$$

$$\tilde{H}_{xx}(\omega) = H_{xx}(\omega) + \frac{\Upsilon_2}{\Sigma_2} H_{xy}(\omega), \quad \tilde{H}_{xy}(\omega) = H_{xy}(\omega),$$

$$\tilde{H}_{yx}(\omega) = H_{yx}(\omega) + \frac{\Upsilon_2}{\Sigma_2} H_{xx}(\omega), \quad \tilde{H}_{yy}(\omega) = H_{yy}(\omega).$$

$$S_{xx}(\omega) = \tilde{H}_{xx}(\omega)\Sigma_2 \tilde{H}^*_{xx}(\omega) + H_{xy}(\omega)\tilde{\Gamma}_2 H^*_{xy}(\omega)$$

where $\tilde{\Gamma}_2 = \Gamma_2 - \frac{\Upsilon_2^2}{\Sigma_2}$.

Unidirectional Causality:

$$f_{Y\to X}(\omega) = \ln \frac{S_{xx}(\omega)}{\tilde{H}_{xx}(\omega)\Sigma_2 \tilde{H}^*_{xx}(\omega)}$$

$$f_{X\to Y}(\omega) = \ln \frac{S_{yy}(\omega)}{\hat{H}_{yy}(\omega)\Gamma_2 \hat{H}^*_{yy}(\omega)}$$

where $\hat{H}_{yy}(\omega) = H_{yy}(\omega) + \frac{\Upsilon_2}{\Gamma_2} H_{yx}(\omega)$.

transformation matrix as $\begin{pmatrix} 1 & -\Upsilon_2/\Gamma_2 \\ 0 & 1 \end{pmatrix}$

Instantaneous Causality:

$$f_{X\cdot Y}(\omega) = \ln \frac{(\tilde{H}_{xx}(\omega)\Sigma_2 \tilde{H}^*_{xx}(\omega))(\hat{H}_{yy}(\omega)\Gamma_2 \hat{H}^*_{yy}(\omega))}{|\mathbf{S}(\omega)|}$$

Example

Consider the following AR(2) model:

$$X_t = 0.9X_{t-1} - 0.5X_{t-2} + \epsilon_t$$

$$Y_t = 0.8Y_{t-1} - 0.5Y_{t-2} + 0.16X_{t-1} - 0.2X_{t-2} + \eta_t$$

$$\varepsilon_t \sim N(0,1), \eta_t \sim N(0\,,0.7), cov(\varepsilon_t, \eta_t) = 0.4$$

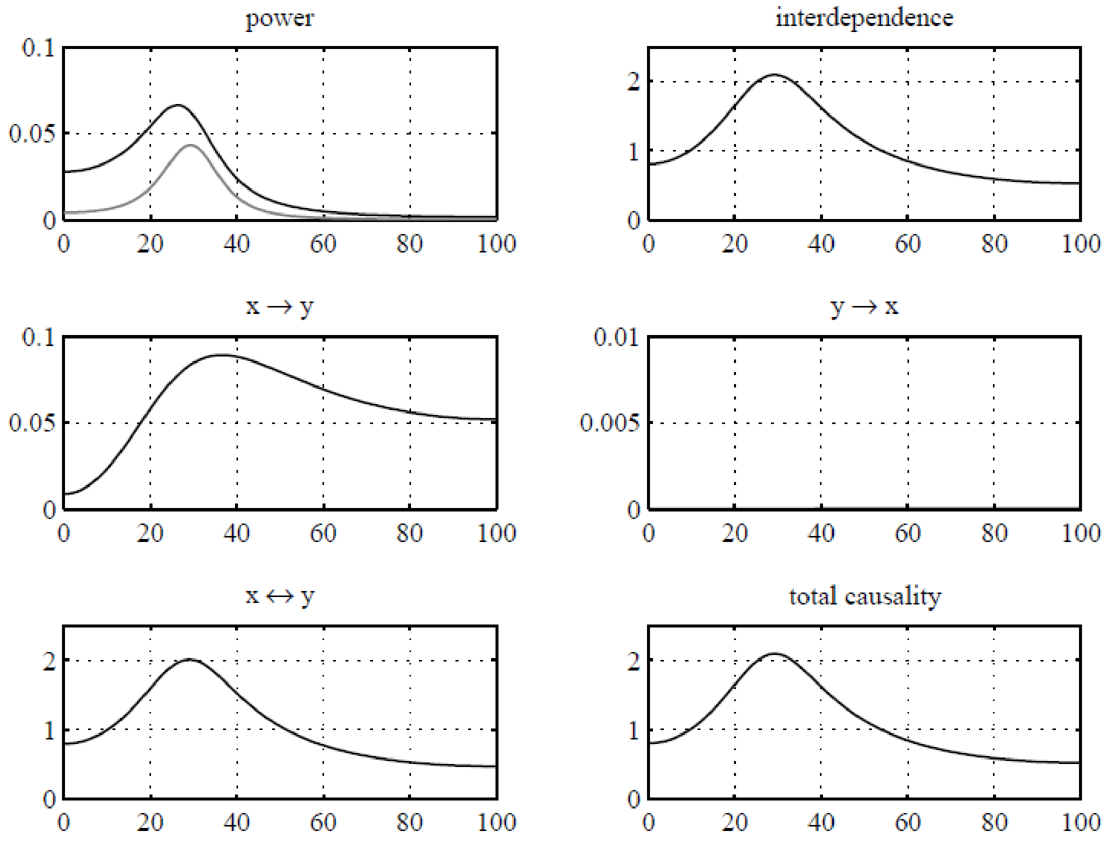


Fig. 2. Simulation results for an AR(2) model consisting of two coupled time series. Power (black for X, gray for Y) spectra, interdependence spectrum (related to the coherence spectrum), and Granger causality spectra are displayed. Note that the total causality spectrum, representing the sum of directional causalities and the instantaneous causality, is nearly identical to the interdependence spectrum.

2.1 Trivariate Granger Causality

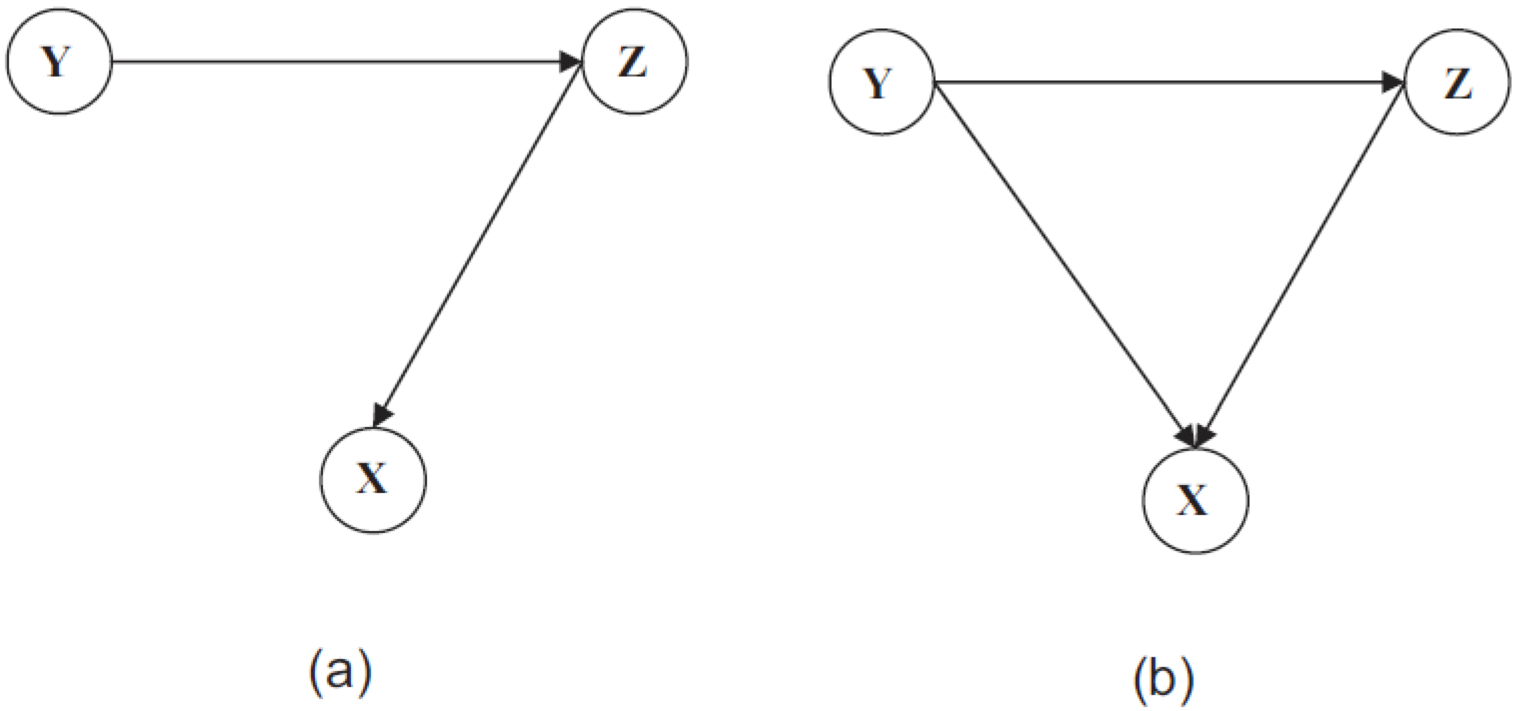
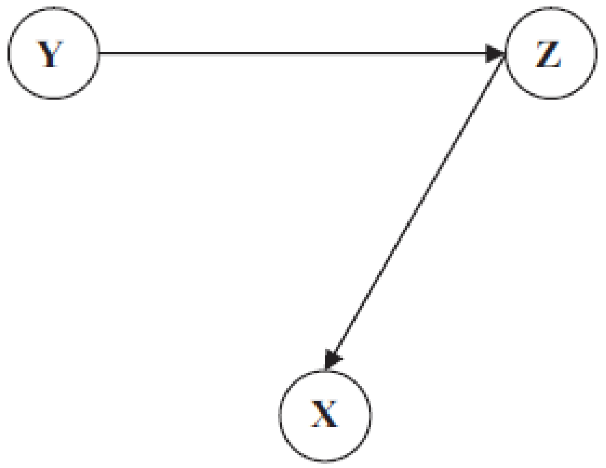


Fig. 1. Two distinct patterns of connectivity among three time series. A pairwise causality analysis cannot distinguish these two patterns.

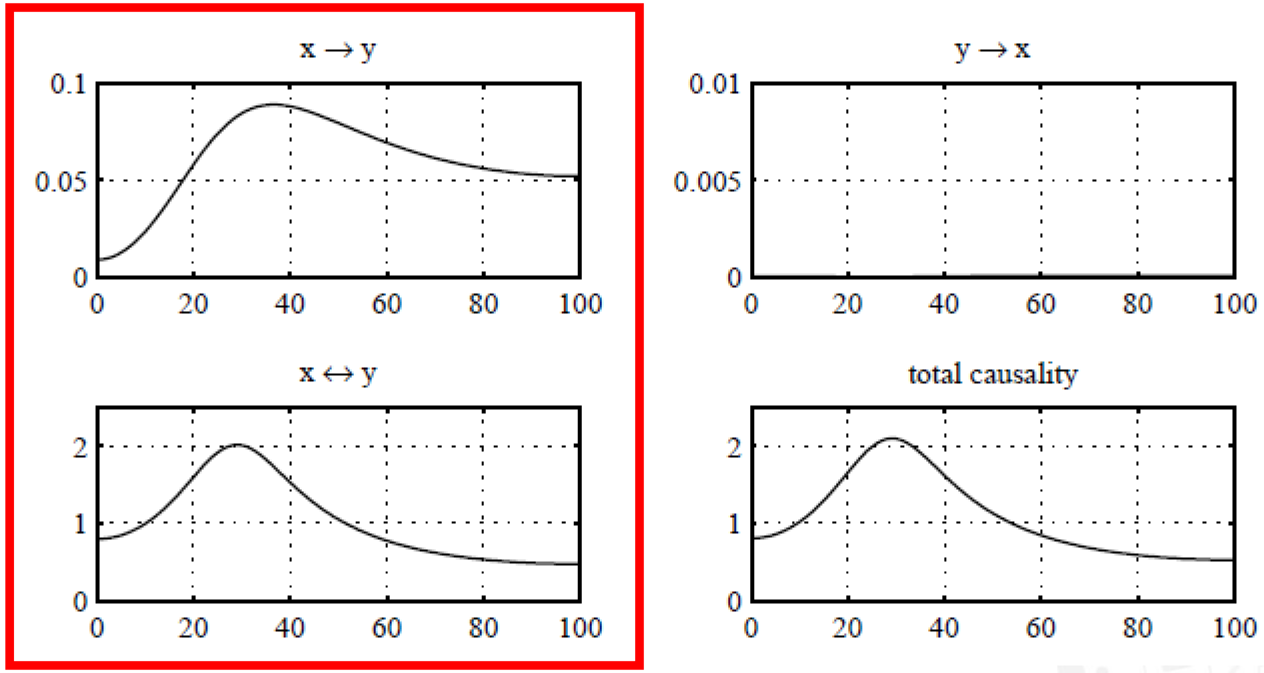
Example 1



$$X_t = 0.8X_{t-1} - 0.5X_{t-2} + 0.4Z_{t-1} + \epsilon_t$$

$$Y_t = 0.9Y_{t-1} - 0.8Y_{t-2} + \xi_t$$

$$Z_t = 0.5Z_{t-1} - 0.2Z_{t-2} + 0.5Y_{t-1} + \eta_t.$$



2.2 Conditional Granger Causality

$$X_t = \sum_{j=1}^{\infty} a_{3j} X_{t-j} + \sum_{j=1}^{\infty} b_{3j} Z_{t-j} + \epsilon_{3t}$$

$$Z_t = \sum_{j=1}^{\infty} c_{3j} X_{t-j} + \sum_{j=1}^{\infty} d_{3j} Z_{t-j} + \gamma_{3t}$$

$$\boldsymbol{\Sigma}_{pair} = \begin{pmatrix} \Sigma_3 & \gamma_3 \\ \gamma_3 & \Gamma_3 \end{pmatrix}$$

Granger Causality from Y_t to X_t conditional on Z_t:

$$F_{Y \to X|Z} = \ln \frac{\Sigma_3}{\Sigma_{xx}}.$$

$$X_t = \sum_{j=1}^{\infty} a_{4j} X_{t-j} + \sum_{j=1}^{\infty} b_{4j} Y_{t-j} + \sum_{j=1}^{\infty} c_{4j} Z_{t-j} + \epsilon_{4t},$$

$$Y_t = \sum_{j=1}^{\infty} d_{4j} X_{t-j} + \sum_{j=1}^{\infty} e_{4j} Y_{t-j} + \sum_{j=1}^{\infty} g_{4j} Z_{t-j} + \eta_{4t},$$

$$Z_t = \sum_{j=1}^{\infty} u_{4j} X_{t-j} + \sum_{j=1}^{\infty} v_{4j} Y_{t-j} + \sum_{j=1}^{\infty} w_{4j} Z_{t-j} + \gamma_{4t},$$

$$\Sigma_4 = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} & \Sigma_{xz} \\ \Sigma_{yx} & \Sigma_{yy} & \Sigma_{yz} \\ \Sigma_{zx} & \Sigma_{zy} & \Sigma_{zz} \end{pmatrix}$$

2.3 Frequency Domain Formulation

$$X_t = \sum_{j=1}^{\infty} a_{3j} X_{t-j} + \sum_{j=1}^{\infty} b_{3j} Z_{t-j} + \epsilon_{3t}$$

$$Z_t = \sum_{j=1}^{\infty} c_{3j} X_{t-j} + \sum_{j=1}^{\infty} d_{3j} Z_{t-j} + \gamma_{3t}$$

$$\begin{pmatrix} X(\omega) \\ Z(\omega) \end{pmatrix} = \begin{pmatrix} G_{xx}(\omega) & G_{xz}(\omega) \\ G_{zx}(\omega) & G_{zz}(\omega) \end{pmatrix} \begin{pmatrix} X^*(\omega) \\ Z^*(\omega) \end{pmatrix}$$

After Fourier Transforming and recasting

$$X_t = \sum_{j=1}^{\infty} a_{4j} X_{t-j} + \sum_{j=1}^{\infty} b_{4j} Y_{t-j} + \sum_{j=1}^{\infty} c_{4j} Z_{t-j} + \epsilon_{4t},$$

$$Y_t = \sum_{j=1}^{\infty} d_{4j} X_{t-j} + \sum_{j=1}^{\infty} e_{4j} Y_{t-j} + \sum_{j=1}^{\infty} g_{4j} Z_{t-j} + \eta_{4t},$$

$$Z_t = \sum_{j=1}^{\infty} u_{4j} X_{t-j} + \sum_{j=1}^{\infty} v_{4j} Y_{t-j} + \sum_{j=1}^{\infty} w_{4j} Z_{t-j} + \gamma_{4t},$$

$$\begin{pmatrix} X(\omega) \\ Y(\omega) \\ Z(\omega) \end{pmatrix} = \begin{pmatrix} H_{xx}(\omega) & H_{xy}(\omega) & H_{xz}(\omega) \\ H_{yx}(\omega) & H_{yy}(\omega) & H_{yz}(\omega) \\ H_{zx}(\omega) & H_{zy}(\omega) & H_{zz}(\omega) \end{pmatrix} \begin{pmatrix} E_x(\omega) \\ E_y(\omega) \\ E_z(\omega) \end{pmatrix}$$

with normalization matrix P = $P_2 * P_1$, where

$$\mathbf{P}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\Sigma_{yx}\Sigma_{xx}^{-1} & 1 & 0 \\ -\Sigma_{zx}\Sigma_{xx}^{-1} & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{P}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(\Sigma_{zy} - \Sigma_{zx}\Sigma_{xx}^{-1}\Sigma_{xy})(\Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy})^{-1} & 1 \end{pmatrix}$$

Useful Property:

$$f_{Y\to X|Z}(\omega) = f_{YZ^*\to X^*}(\omega).$$

Error terms for X and Z

$$\begin{pmatrix} X(\omega) \\ Z(\omega) \end{pmatrix} = \begin{pmatrix} G_{xx}(\omega) & G_{xz}(\omega) \\ G_{zx}(\omega) & G_{zz}(\omega) \end{pmatrix} \begin{pmatrix} X^*(\omega) \\ Z^*(\omega) \end{pmatrix}$$

$$\begin{pmatrix} X(\omega) \\ Y(\omega) \\ Z(\omega) \end{pmatrix} = \begin{pmatrix} H_{xx}(\omega) & H_{xy}(\omega) & H_{xz}(\omega) \\ H_{yx}(\omega) & H_{yy}(\omega) & H_{yz}(\omega) \\ H_{zx}(\omega) & H_{zy}(\omega) & H_{zz}(\omega) \end{pmatrix} \begin{pmatrix} E_x(\omega) \\ E_y(\omega) \\ E_z(\omega) \end{pmatrix}$$

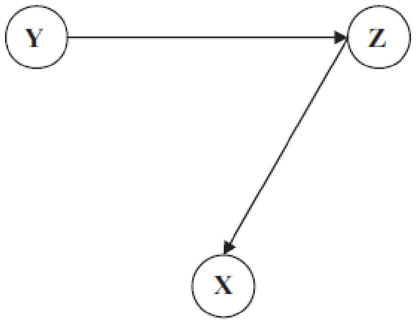
$$\begin{aligned} \begin{pmatrix} X^*(\omega) \\ Y(\omega) \\ Z^*(\omega) \end{pmatrix} &= \begin{pmatrix} G_{xx}(\omega) & 0 & G_{xz}(\omega) \\ 0 & 1 & 0 \\ G_{zx}(\omega) & 0 & G_{zz}(\omega) \end{pmatrix}^{-1} \begin{pmatrix} H_{xx}(\omega) & H_{xy}(\omega) & H_{xz}(\omega) \\ H_{yx}(\omega) & H_{yy}(\omega) & H_{yz}(\omega) \\ H_{zx}(\omega) & H_{zy}(\omega) & H_{zz}(\omega) \end{pmatrix} \begin{pmatrix} E_x(\omega) \\ E_y(\omega) \\ E_z(\omega) \end{pmatrix} \\ &= \begin{pmatrix} Q_{xx}(\omega) & Q_{xy}(\omega) & Q_{xz}(\omega) \\ Q_{yx}(\omega) & Q_{yy}(\omega) & Q_{yz}(\omega) \\ Q_{zx}(\omega) & Q_{zy}(\omega) & Q_{zz}(\omega) \end{pmatrix} \begin{pmatrix} E_x(\omega) \\ E_y(\omega) \\ E_z(\omega) \end{pmatrix}, \end{aligned} \quad (42)$$

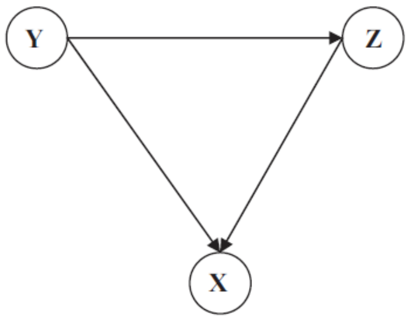
$$S_{x^*x^*}(\omega) = Q_{xx}(\omega)\hat{\Sigma}_{xx}Q^*_{xx}(\omega) + Q_{xy}(\omega)\hat{\Sigma}_{yy}Q^*_{xy}(\omega) + Q_{xz}(\omega)\hat{\Sigma}_{zz}Q^*_{xz}(\omega).$$

$$f_{YZ^*\to X^*}(\omega) = \ln \frac{|S_{x^*x^*}(\omega)|}{|Q_{xx}(\omega)\hat{\Sigma}_{xx}Q^*_{xx}(\omega)|}$$

$$f_{Y\to X|Z}(\omega) = \ln \frac{\Sigma_3}{|Q_{xx}(\omega)\hat{\Sigma}_{xx}Q^*_{xx}(\omega)|}.$$

Example 2

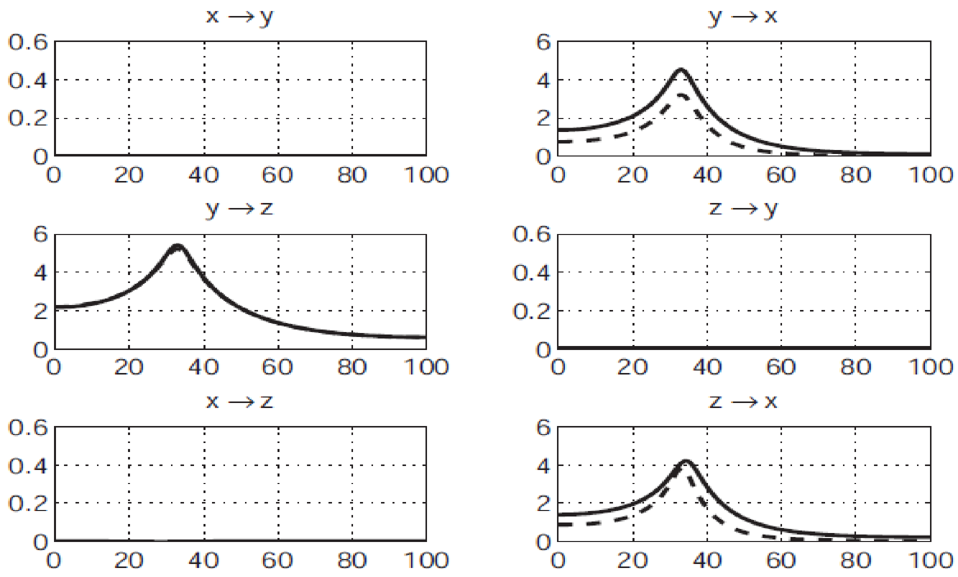


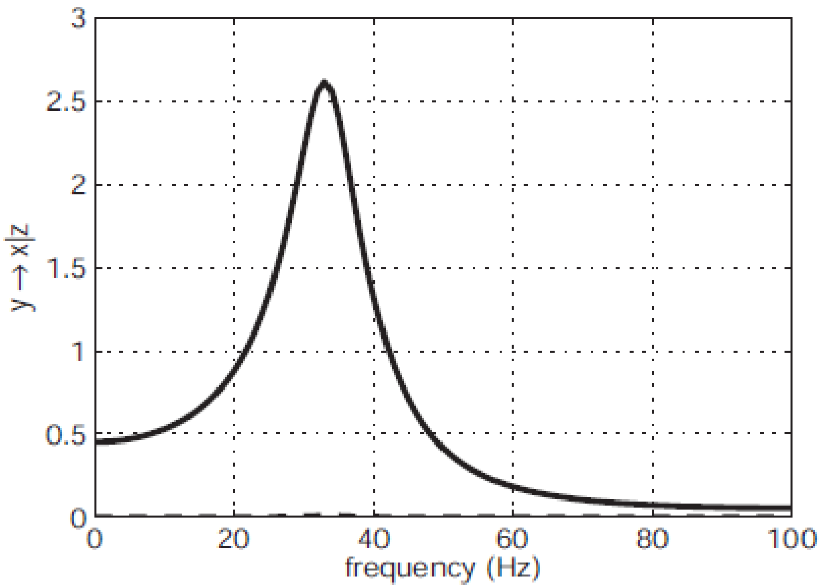


$$X_t = 0.8X_{t-1} - 0.5X_{t-2} + 0.4Z_{t-1} + 0.2Y_{t-2} + \epsilon_t$$

$$Y_t = 0.9Y_{t-1} - 0.8Y_{t-2} + \xi_t$$

$$Z_t = 0.5Z_{t-1} - 0.2Z_{t-2} + 0.5Y_{t-1} + \eta_t.$$





(b)

Thank you!