Learning From Data
Lecture 11: Reinforcement Learning

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TBSI

December 8, 2022

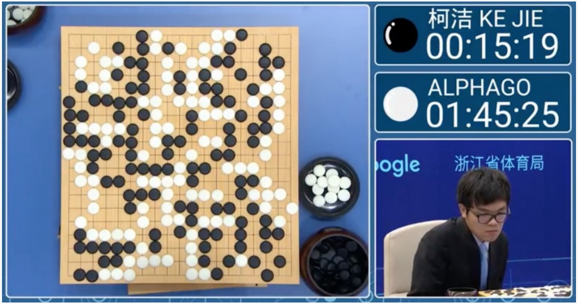
Today's Lecture

Reinforcement Learning

- What's reinforcement learning?
- Mathematical formulation: Markov Decision Process (MDP)
- Model Learning for MDP, Fitted Value Iteration
- Deep reinforcement learning (Deep Q-networks)

Deep Reinforcement Learning: AlphaGo

AlphaGo beat World Go Champion Kejie (2017)



Nature paper on by AlphaGo team

Deep Reinforcement Learning: OpenAI

OpenAI beats Dota2 world champion (2017)

Elon Musk @elonmusk

Follow

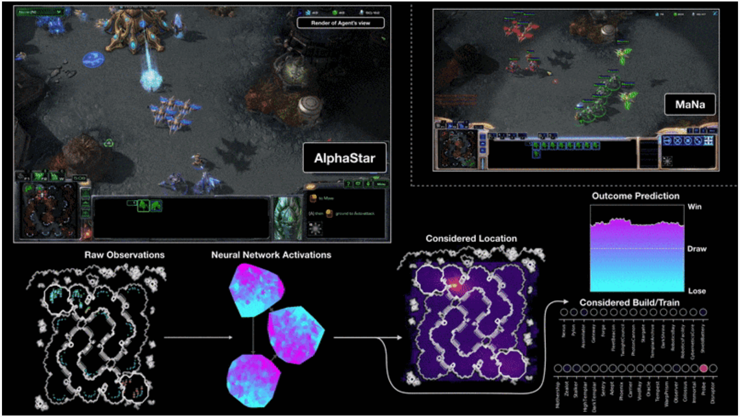
OpenAI first ever to defeat world's best players in competitive eSports. Vastly more complex than traditional board games like chess & Go.

3:15 AM - Aug 12, 2017

647 6,818 23,006

Multi-Agent Reinforcement Learning: AlphaStar

AlphaStar reached Grandmaster level in StarCraft II (2019)



https://www.nature.com/articles/s41586-019-1724-z

Reinforcement Learning: Autonomous Car, Helicopter

Stanley, Winner of DARPA Grand Challenge (2005)
Inverted autonomous helicopter flight (2004)

Other applications include robotic control, computational economics, health care...

What is reinforcement learning?

Sequential decision making

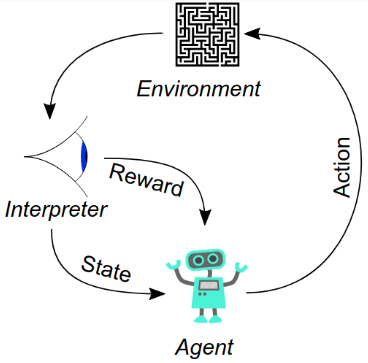
To deciding, from **experience**, the **sequence of actions** to perform in an **uncertain environment** in order to achieve some **goals**.

- e.g. play games, robotic control, autonomous driving, smart grid
- Do not have full knowledge of the environment a prior
- Difficult to label a sample as "the right answer" for a learning goal

What is reinforcement learning?

A learning framework to solve sequential decision making problem, inspired by behavior psychology (Sutton, 1984)

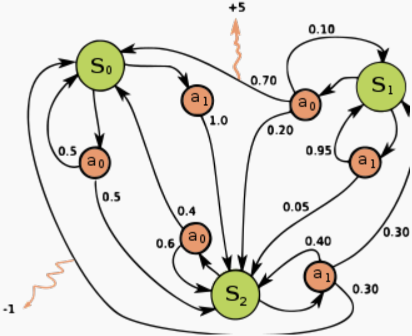
- An agent interacts with an environment which provides a "reward function" to indicate how "well" the learning agent is doing
- The agents take actions to maximize the cumulative "reward"



Markov Decision Process

A Markov decision process $(S, A, \{P_{sa}\}, \gamma, R)$

- S: a set of **states** (environment)
- A: a set of **actions**
- $P_{sa} := P(s_{t+1}|s_t, a_t)$: **state transition** probabilities. *Markov property:* $P(s_{t+1}|s_t, a_t) = P(s_{t+1}|s_t, a_t, \ldots, s_0, a_0)$.
- $R : S \times A \to \mathbb{R}$ is a **reward function**
- $\gamma \in [0, 1)$: discount factor



$S = \{S_0, S_1, S_2\}$

$A = \{a_0, a_1\}$

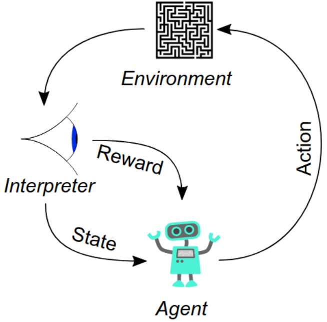
$R(s_1, a_0) = 5, R(s_2, a_1) = -1$

	S_0	S_1	S_2
S_0, a_0	0.5	0	0.5
S_0, a_1	0	0	1
S_1, a_0	0.7	0.1	0.2
S_1, a_1	0	0.95	0.05
S_2, a_0	0.4	0.6	0
S_2, a_1	0.3	0.3	0.4

Markov Decision Process: Overview

At time step $t = 0$ with initial state $s_0 \in S$
for $t = 0$ until done:

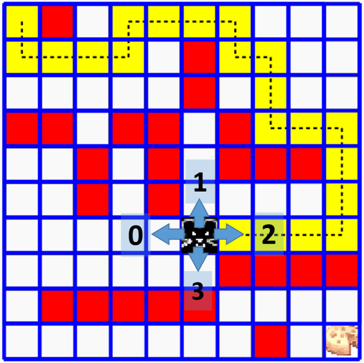
- Agent selects action at $a_t \in A$
- Environment yields reward $r_t = R(s_t, a_t)$
- Environment samples next state $s_{t+1} \sim P_{sa}$
- Agent receives reward r_t and next state s_{t+1}



A **policy** $\pi : S \to A$ specifies what action to take in each state

Goal: find optimal policy π^* that maximizes cumulative discounted reward

MDP Example: Maze Solver



https://www.samyzaf.com/ML/rl/qmaze.html

Goal: get to the bottom-right corner of the nxn maze

- S: position of the agent (mouse)
- A: $\{\text{Left, Right, Up, Down}\}$
- $P_{sa}(s') = \begin{cases} 1 & s' \text{ is next move} \\ 0 & \text{otherwise} \end{cases}$
- $R(a, s) =$
$$\begin{cases} -0.05 & \text{move to free cell} \\ -1 & \text{move to wall/block} \\ 1 & \text{move to goal} \end{cases}$$
- $\gamma \in [0, 1)$: discount factor

MDP Example: Maze Solver

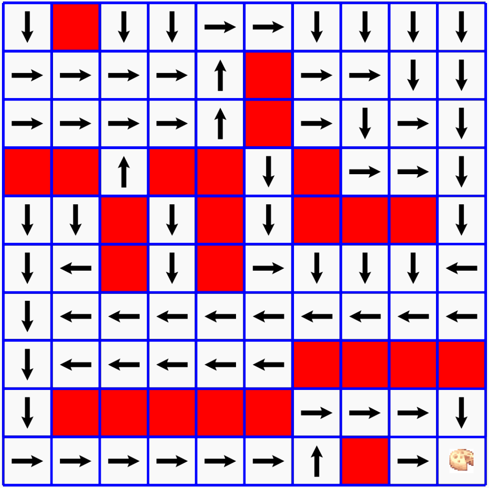
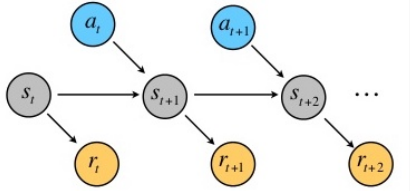


Figure: An optimal policy function $\pi(s)$ learned by the solver.

https://www.samyzaf.com/ML/rl/qmaze.html

Markov Decision Process

Consider a sequence of states $s_0, s_1, \dots$ with actions $a_0, a_1, \dots,$



Total payoff of a sequence:

$$R(s_0, a_0) + \gamma R(s_1, a_1) + \gamma^2 R(s_2, a_2) + \dots$$

For simplicity, let's assume rewards only depends on state s, i.e.

$$R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \dots$$

Future reward at step t is discounted by γ^t

Policy & value functions

Goal of reinforcement learning: choose actions that maximize the expected total payoff

$$\mathbb{E}[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \dots]$$

A **policy** is any function $\pi \colon S \to A$.
A **value function** of policy π is the expected payoff if we start from s, take actions according to π

$$V^\pi(s) = \mathbb{E}[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \dots | s_0 = s, \pi]$$

Given π, value function satisfies the **Bellman equation**: *Why?*

$$V^\pi(s) = R(s) + \gamma \sum_{s' \in S} P_{s\pi(s)}(s') V^\pi(s')$$

Bellman Equation

Value function of π at s:

$$V^{\pi}(s) = \mathbb{E}[R(s_0, \pi(s_0) + \gamma R(s_1, \pi(s_1)) + \gamma^2 R(s_2, \pi(s_2)) + \ldots | s_0 = s, \pi]$$

Assume action is known:

$$\begin{aligned} V^{\pi}(s) &= \mathbb{E}[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots | s_0 = s, \pi] \\ &= \mathbb{E}[R(s)] + \gamma \mathbb{E}[R(s_1) + \gamma R(s_2) + \ldots | s_0 = s, \pi] \\ &= R(s) + \gamma \mathbb{E}_{s' \sim P_{s,\pi(s)}}[V^{\pi}(s')] \\ &= R(s) + \gamma \sum_{s' \in S} P_{s,\pi(s)}(s') V^{\pi}(s') \\ &\text{or } R(s) + \gamma \int_{s'} P_{s,\pi(s)}(s') V^{\pi}(s') ds' \end{aligned}$$

$V^{\pi}(s)$ can be solved as $|S|$ linear equations with $|S|$ unknowns.

Optimal value and policy

We define the **optimal value function**

$$V^*(s) = \max_{\pi} V^{\pi}(s) = R(s) + \max_{\pi} \gamma \sum_{s' \in S} P_{s,\pi(s)}(s') V^{\pi}(s')$$
$$= R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s') V^*(s')$$

Let $\pi^* : S \to A$ be the policy that attains the 'max':

$$\pi^*(s) = \operatorname*{argmax}_{a \in A} \sum_{s' \in S} P_{sa}(s') V^*(s')$$

Then for every state s and every policy π, we can show

$$V^*(s) = V^{\pi^*}(s) \geq V^{\pi}(s)$$

π^* is the optimal policy for any initial state s

Optimal value and policy

Proposition 1

For every state s and every policy π,

$$V^*(s) = V^{\pi^*}(s)$$

Proof. $\square$

Solving finite-state MDP: value iteration

Assume the MDP has finite state and action space.

```
1. For each state s, initialize V(s) := 0
2. Repeat until convergence {
        Update V(s) := R(s) + max_{a∈A} γ Σ_{s'∈S} P_sa(s') V(s')
          for every state s
   }
```

Two ways to update $V(s)$:

- Synchronous update:

```
Set V_0(s) := V(s)  for all states s ∈ S
For each s ∈ S:
      V(s) := R(s) + max_{a∈A} γ Σ_{s'∈S} P_sa(s') V_0(s')
```

- Asynchronous update:

```
For each s ∈ S:
      V(s) := R(s) + max_{a∈A} γ Σ_{s'∈S} P_sa(s') V(s')
```

Solving finite-state MDP: policy iteration

1. Initialize π randomly
2. Repeat until convergence {
    a. Let $V := V^{\pi}$
    b. For each state s,
        $\pi(s) := \operatorname{argmax}_{a \in A} \sum_{s'} P_{sa}(s') V(s')$
  }

Step (a) can be done by solving Bellman's equation.

Discussion

Both value iteration and policy iteration will converge to V^* and π^*

Value iteration vs. policy iteration

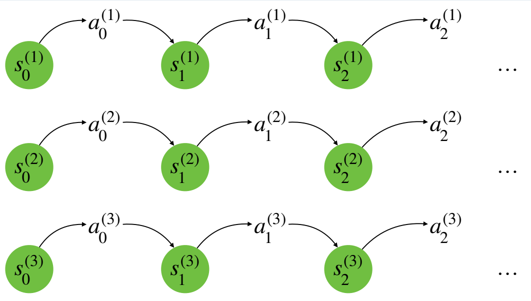
- Policy iteration is more efficient and converge faster for small MDP
- Value iteration is more practical for MDP's with large state spaces

Learning a model for finite-state MDP

Suppose the reward function $R(s)$ and the transition probability P_{sa} is not known. How to estimate them from data?

Experience from MDP

Given policy π, execute π repeatedly in the environment:



$\vdots$

Estimate model from experience

Estimate P_{sa}

Maximum likelihood estimate of state transition probability:

$$P_{sa}(s') = P(s'|s,a) = \frac{\#\{s \xrightarrow{a} s'\}}{\#\{s \xrightarrow{a} \cdot\}}$$

If $\#\{s \xrightarrow{a} \cdot\} = 0$, set $P_{sa}(s') = \frac{1}{|S|}$.

Estimate $R(s)$

Let $R(s)^{(t)}$ be the immediate reward of state s in the t-th trail,

$$R(s) = \mathbb{E}[R(s)^{(t)}] = \frac{1}{m}\sum_{t=1}^{m} R(s)^{(t)}$$

Algorithm: MDP Model Learning

1. Initialize π randomly, $V(s) := 0$ for all s
2. Repeat until convergence {
   a. Execute π for m trails
   b. Update P_{sa} and R using the accumulated experience
   c. $V :=$ ValueIteration(P_{sa}, R, V)
   b. Update π greedily with respect to V:
      $\pi(s) := \text{argmax}_{a \in A} \sum_{s'} P_{sa}(s') V(s')$
   }

ValueIteration(P_{sa}, R, V_0)

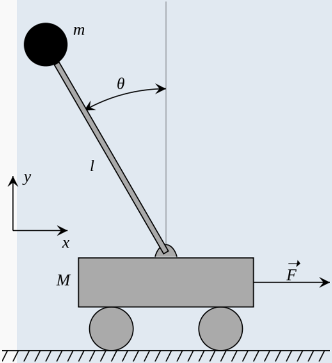
1. Initialize $V = V_0$
2. Repeat until convergence {
      Update $V(s) := R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s') V(s')$
      for every state s
   }

Continuous state MDPs

An MDP may have an infinite number of states:

- A car's state : $(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$
- A helicopter's state : $(x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, \dot{\phi}, \dot{\theta}, \dot{\psi})$

1D Inverted Pendulum



Control goal: balance the pole on the cart

- State representation: $(x, \theta, \dot{x}, \dot{\theta})$
- Action: force F on the car
- Reward: $+1$ each time the pole is upright

Due to the Curse of Dimensionality, discretization rarely works well in continuous state with more than 1-2 dimensions

Value function approximation

How to approximate V directly without resorting to discretization?

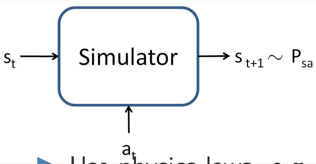
Main ideas:

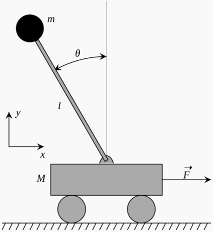
- Obtain a *model* or *simulator* for the MDP, to produce **experience tuples**: $\langle s, a, s', r\rangle$
- Sample $s^{(1)}, \ldots, s^{(m)}$ from the state space S, estimate their optimal expected total payoff using the model, i.e. $y^{(1)} \approx V(s^{(1)}), y^{(2)} \approx V(s^{(2)}), \ldots$
- Approximate V as a function of state s using supervised learning from $(s^{(1)}, y^{(1)}), (s^{(2)}, y^{(2)}), \ldots$ e.g.

$$V(s) = \theta^T \phi(s)$$

Obtaining a simulator

A **simulator** is a black box that generates the next state s_{t+1} given current state s_t and action a_t.





- Use physics laws. e.g. equation of motion for the inversed pendulum problem:

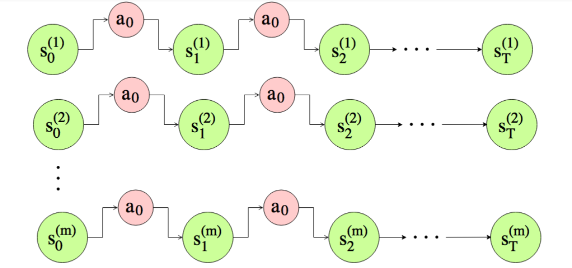
$$(m+M)\ddot{x} + mL(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos(\theta)) = F$$

$$g\sin\theta + \ddot{x}\cos\theta = L\ddot{\theta}$$

- Use out-of-the-shelf simulation software
- Game simulator

Obtaining a model from data

Execute m trails in which we repeatedly take actions in an MDP, each trial for T timesteps.



Learn a prediction model $s_{t+1} = h_\theta \left(\begin{bmatrix} s_t \\ a_t \end{bmatrix} \right)$ by picking

$$\theta^* = \underset{\theta}{\text{argmin}} \sum_{i=1}^{m} \sum_{t=0}^{T-1} \left\| s_{t+1}^{(i)} - h_\theta \left(\begin{bmatrix} s_t^{(i)} \\ a_t^{(i)} \end{bmatrix} \right) \right\|^2$$

Obtaining a model from data

Popular prediction models

- Linear function: $h_\theta = As_t + Ba_t$
- Linear function with feature mapping: $h_\theta = A\phi_s(s_t) + B\phi_a(a_t)$
- Neural network

Build a simulator using the model:

- Deterministic model: $s_{t+1} = h_\theta\left(\begin{bmatrix} s_t \\ a_t \end{bmatrix}\right)$
- Stochastic model: $s_{t+1} = h_\theta\left(\begin{bmatrix} s_t \\ a_t \end{bmatrix}\right) + \epsilon_t \ , \ \ \epsilon_t \sim \mathcal{N}(0, \Sigma)$

Value function approximation

How to approximate V directly without resorting to discretization?

Main ideas:

- Obtain a model or simulator for the MDP
- Sample $s^{(1)}, \dots, s^{(m)}$ from the state space S, estimate their optimal expected total payoff using the model, i.e. $y^{(1)} \approx V(s^{(1)}), y^{(2)} \approx V(s^{(2)}), \dots$
- Approximate V as a function of state s using supervised learning from $(s^{(1)}, y^{(1)}), (s^{(2)}, y^{(2)}), \dots$ e.g.

$$V(s) = \theta^T \phi(s)$$

Value function for continuous states

Update for finite-state value function:

$$V(s) := R(s) + \gamma \max_{a \in A} \sum_{s' \in S} P_{sa}(s') V(s')$$

Update we want for continuous-state value function:

$$\begin{aligned} V(s) &:= R(s) + \gamma \max_{a \in A} \int_{s'} P_{sa}(s') V(s') ds' \\ &= R(s) + \gamma \max_{a \in A} \mathbb{E}_{s' \sim P_{sa}} [V(s')] \end{aligned}$$

For each sample state s, we compute $y^{(i)}$ to approximate $R(s) + \gamma \max_{a \in A} \mathbb{E}_{s' \sim P_{s^{(i)}a}}[V(s')]$ using finite samples from P_{sa}

Value function approximation

How to approximate V directly without resorting to discretization?

Main ideas:

- Obtain a model or simulator for the MDP
- Sample $s^{(1)}, \dots, s^{(m)}$ from the state space S, estimate their optimal expected total payoff using the model, i.e. $y^{(1)} \approx V(s^{(1)}), y^{(2)} \approx V(s^{(2)}), \dots$
- Approximate V as a function of state s using supervised learning from $(s^{(1)}, y^{(1)}), (s^{(2)}, y^{(2)}), \dots$ e.g.

$$V(s) = \theta^T \phi(s)$$

Fitted value iteration

Algorithm: Fitted value iteration (Stochastic Model)

1. Sample $s^{(1)}, \ldots, s^{(m)} \in S$
2. Initialize $\theta := 0$
2. Repeat {
   a. For each sample $s^{(i)}$
      For each action a:
         Sample $s'_1, \ldots, s'_k \sim P_{s^{(i)},a}$ using a model
         Compute $Q(a) = \frac{1}{k}\sum_{j=1}^{k} R(s^{(i)}) + \gamma V(s'_j)$
         ↑ estimates $R(s^{(i)}) + \gamma \mathbb{E}_{s' \sim P_{s'a}}[V(s')]$
         where $V(s) := \theta^T \phi(s)$

      $y^{(i)} = \max_a Q(a)$
      ↑ estimates $R(s^{(i)}) + \gamma \max_a \mathbb{E}_{s' \sim P_{s'a}}[V(s')]$
   b. Update θ using supervised learning:
      $\theta := \mathrm{argmin}_\theta \frac{1}{2}\sum_{i=1}^{m}(\theta^T \phi(s^{(i)}) - y^{(i)})^2$
   }

If the model is deterministic, set $k = 1$

Computing the optimal policy

After obtaining the value function approximation V, the corresponding policy is

$$\pi(s) = \underset{a}{\operatorname{argmax}} \, \mathbb{E}_{s' \sim P_{sa}}[V(s')])$$

Estimate the optimal policy from experience:

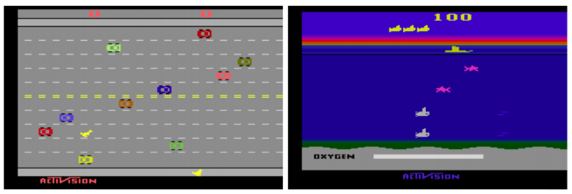
```
For each action a :
   1. Sample s'_1, ..., s'_k ~ P_{s,a} using a model
   2. Compute Q(a) = 1/k Σ_{j=1}^k R(s) + γV(s'_j)
π(s) = argmax_a Q(a)
```

Instead of linear regression, other learning algorithms can be used to estimate $V(s)$.

Two Outstanding Success Stories

Atari AI [Minh et al. 2015]

- Plays a variety of Atari 2600 video games at superhuman level
- Trained directly from image pixels, based on a single reward signal



AlphaGo [Silver et al. 2016]

- A hybrid deep RL system
- Trained using supervised and reinforcement learning, in combination with a traditional tree-search algorithm.

Deep Reinforcement Learning

Main difference from classic RL:

- Use deep network to represent value function
- Optimize value function end-to-end
- Use stochastic gradient descent

Q-Value Function

Given policy π which produce sample sequence $(s_0, a_0, r_0), (s_1, a_1, r_1), \ldots$

- Value function of π :

$$V^{\pi}(s) = \mathbb{E}\left[\sum_{t\geq 0} \gamma^t r_t \middle| s_0 = s, \pi\right]$$

- The **Q-value function** $Q^{\pi}(s, a)$ is the expected payoff if we take a at state s and follow π

$$Q^{\pi}(s, a) = \mathbb{E}\left[\sum_{t\geq 0} \gamma^t r_t \middle| s_0 = s, a_0 = a, \pi\right]$$

- The optimal Q-value function is:

$$Q^{*}(s, a) = \max_{\pi} Q^{\pi}(s, a) = \max_{\pi} \mathbb{E}\left[\sum_{t\geq 0} \gamma^t r_t \middle| s_0 = s, a_0 = a, \pi\right]$$

Q-Learning

Bellman's equation for Q-Value function:

$$Q^*(s,a) = \mathbb{E}_{s' \sim \mathcal{E}}[r + \gamma \max_{a'} Q^*(s',a') | s,a]$$

Value iteration is not practical when the search space is large.

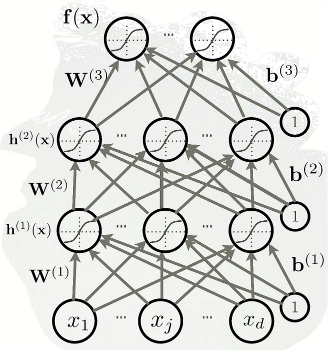
e.g. In an Atari game, each frame is an 128-color 210×160 image, then $|S| = 128^{210 \times 160}$

- Uses a function approximation:

$$Q(s,a;\theta) \approx Q^*(s,a)$$

- In deep Q-learning, $Q(s,a;\theta)$ is a neural network

Neural Network Review



Training goal: $\min_\theta \sum_{i=1}^{m} L(f(x^{(i)};\theta), y^{(i)})$

Forward propagation

Initialize $h^{(0)}(x) = x$

For each layer $l = 1 \ldots d$:

- $a^{(l)}(x) = W^{(l)} h^{(l-1)}(x) + b^{(l)}$
- $h^{(l)}(x) = g(a^{(l)}(x))$

Evaluate loss function $L(h^{(d)}(x), y)$

Backward propagation

Compute gradient $\frac{dL}{dh^{(d)}}$

For each layer $l = d \ldots 1$:

- Update gradient for parameters in layer l

Q-Networks

Training goal: find $Q(s, a; \theta)$ that fits Bellman's equation:
$Q^*(s, a) = \mathbb{E}_{s' \sim \mathcal{E}}[r + \gamma \max_{a'} Q^*(s', a')|s, a]$

Forward Pass

Loss function:

$$L_i(\theta_i) = \mathbb{E}_{s,a}[(y_i - Q(s, a; \theta_i)^2]$$

where $y_i = \mathbb{E}_{s' \sim \mathcal{E}}[r + \gamma \max_{a'} Q(s', a'; \theta_{i-1})|s, a]$

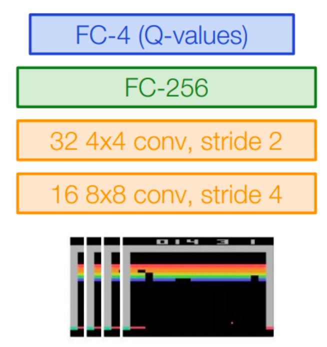
Backward Pass

Update parameter θ by computing gradient

$$\nabla_{\theta_i} L_i(\theta_i) = \mathbb{E}_{s,a,s' \sim \mathcal{E}} \left[\left(r + \gamma \max_{a'} Q(s', a'; \theta_{i-1}) - Q(s, a; \theta_i) \right) \nabla_\theta Q(s, a; \theta_i) \right]$$

Deep Q-Network Architecture

- Input: 4 consecutive frames
- Preprocessing: convert to grayscale, down-sampling, cropping. Final dimension $84 \times 84 \times 4$
- Output: Q-value functions for 4 actions $Q(s, a_1)$, $Q(s, a_2)$, $Q(s, a_3)$, $Q(s, a_4)$



Experience Replay

Challenge of standard deep Q-learning: correlated input

- invalidate the i.i.d. assumption on training samples
- current policy may restrict action samples we experience in the environment

Experience replay

- Store past transitions (s_t, a_t, r_t, s_{t+1}) within a sliding window in the **replay memory** D.
- Train Q-Network using random mini-batch sampled from D to reduce sample correlation
- Also reduces total running time by reusing samples

The Algorithm

Algorithm 1 Deep Q-learning with Experience Replay

Initialize replay memory $\mathcal{D}$ to capacity N
Initialize action-value function Q with random weights
for episode $= 1, M$ **do**
  Initialise sequence $s_1 = \{x_1\}$ and preprocessed sequenced $\phi_1 = \phi(s_1)$
  for $t = 1, T$ **do**
    With probability ϵ select a random action a_t
    otherwise select $a_t = \max_a Q^*(\phi(s_t), a; \theta)$
    Execute action a_t in emulator and observe reward r_t and image x_{t+1}
    Set $s_{t+1} = s_t, a_t, x_{t+1}$ and preprocess $\phi_{t+1} = \phi(s_{t+1})$
    Store transition $(\phi_t, a_t, r_t, \phi_{t+1})$ in $\mathcal{D}$
    Sample random minibatch of transitions $(\phi_j, a_j, r_j, \phi_{j+1})$ from $\mathcal{D}$
    Set $y_j = \begin{cases} r_j & \text{for terminal } \phi_{j+1} \\ r_j + \gamma \max_{a'} Q(\phi_{j+1}, a'; \theta) & \text{for non-terminal } \phi_{j+1} \end{cases}$
    Perform a gradient descent step on $(y_j - Q(\phi_j, a_j; \theta))^2$ according to equation 3
  end for
end for

Parameter ϵ controls the exploration vs. optimization trade-off

Reinforcement Learning Demo

See Demo.
https://cs.stanford.edu/people/karpathy/convnetjs/demo/rldemo.html