VC Dimension:

target: $f: x \to y$ function / distribution.

↑ optimization, loss + GD ↑ Prb / MLE. MAP.

H: hypothesis set. (candidate h) $\Rightarrow$ g = best $h \approx f$

e.g. perceptron:

$$h(x) = \text{sign}\left(\sum_i w_i x_i - b\right)$$

$$f \approx g : w_g \Leftarrow \underset{w}{\arg\min} \underbrace{\sum \left[y_n \neq \text{sign}(w^T x) \right]}_{\text{loss}}$$

ERM: Hoeffding inequality equation:

$$p(|\nu - \mu| > \varepsilon) \le e^{-2\varepsilon^2 N}.$$

$$\downarrow \quad p(|\varepsilon(h) - \hat{\varepsilon}(h)| > \varepsilon) \le 2e^{-2\varepsilon^2 N}. \quad \forall h \in H.$$

$$P(|\varepsilon(g) - \hat{\varepsilon}(g)| > \varepsilon) \le 2 \cdot M \cdot e^{-2\varepsilon^2 N}.$$

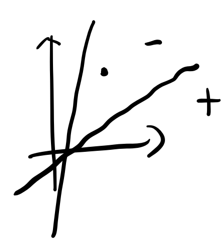
$M = |H| \to$ infinite.

LTF / in $\mathbb{R}^2$.

$M \downarrow m(H) \downarrow$ VCD.

VC D: limited h classes.

LTF



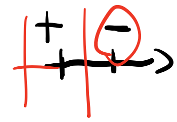
① label x +

② label x −.

dichotomies. $\le 2^N$

Examples of VCD. shatter / break point.

① VC (Postive Half Lines) = 1 , $\mathcal{X} = \mathbb{R}$.



$h(x) = \text{sign}(x-b)$

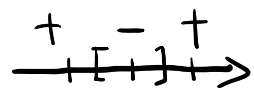
VC < 2 : $\forall\, x_1 < x_2 \in \mathbb{R}$ construct $y_1 \cdot y_2$. $y_2 = -1$, $y_1 = 1$

to show that $h(x_i) \neq y_i$

VC ≥ 1 : $\exists\, x_1 . x_2$ $\forall\, y_1 y_2$. $h(x_i) = y_i$

$\begin{cases} x_1 < x_2 \quad y_2 = -1 \quad y_1 = 1 \\ \text{sign}(x_1 - b) = 1 \Rightarrow x_1 > b \\ \text{sign}(x_2 - b) = -1 \Rightarrow x_2 < b \end{cases}$

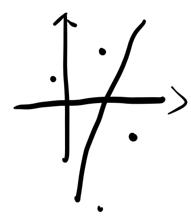
② VC (Intervals) = 2.



VC < 3 : $\forall\, x_1 < x_2 < x_3 \in \mathbb{R}$ $y_1 = 1$. $y_2 = -1$ $y_3 = 1$

$h(x) = \mathbb{1}(a < x < b)$ $\forall\, a.b.$ $h(x_2) \neq y_2$.

③ VC (LTF in $\mathbb{R}^2$ / $\mathbb{R}^n$.) = 3



$h(x) = \text{sign}(w^T x + b)$.

VC (LTF in $\mathbb{R}^n$) = n+1 , $\mathcal{X} = \mathbb{R}^n$. ?

① VC ≥ n+1 :

Find $x_i \in \mathbb{R}^n$ $(x_1, x_2, \cdots, x_n, x_{n+1})$

for any $\forall$ $(y_1, y_2, \cdots y_n, y_{n+1})$

$h(x_i) = y_i$.

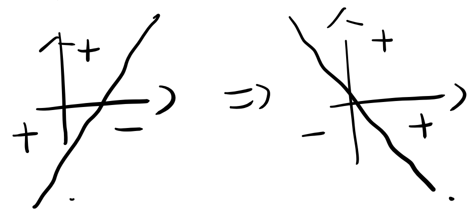
$b = y_{n+1}$ $x_{n+1} = 0$ $w^T x = \sum w_i x_i = \sum_i (y_k - y_{n+1}) x_i^T$

$x_1 \cdots x_n$ $x_i^T x_j = 0$ if $i \neq j$.

$h(x_k) = \text{sign}\big(\underline{\sum_i (y_k - y_{n+1}) x_i^T \cdot x_k} + \underline{y_{n+1}}\big) = y_k$.

$$(y_k - y_{n+1}) \cdot x_k^T \cdot x_k = 1$$

h^* given $\underline{(y_1 \cdots y_n)}$



② $VC < n+2$. $\forall\ (x_1, x_2, \cdots, x_{n+1}, x_{n+2})$

construct $Y^* = (y_1, \cdots\ y_{n+1},\ y_{n+2})$

s.t. $h(x_i) \neq y_i$. for some i.

$h(x_i) = y_i$

$$\begin{cases} w^T x_1 + b = y_1 \\ w^T x_2 + b = y_2 \\ \vdots \\ w^T x_{n+1} + b = y_{n+1} \end{cases}$$

$w^T x_{n+2} + b = y_{n+2}$. $\leftarrow$

$x \in \mathbb{R}^n$.

$n+1$ independent.

$n+2$

$$x_{n+2} = \sum_{i=1}^{n+1} k_i x_i$$

$$y_{n+2} = \underline{\sum_{i=1}^{n+1} k_i'}\ \underline{y_i}.$$

$\neq$

details:

$\mathrm{sign}(w^T x_0 + b) = y_0$

$$\begin{cases} \mathrm{sign}(w^T x_1 + b) = y_1 \\ \vdots \\ \mathrm{sign}(w^T x_{n+1} + b) = y_{n+1} \end{cases}$$

$x_{n+1} = \sum_{i=1}^{n} k_i x_i$ when $\{x_i\}$ form a basis

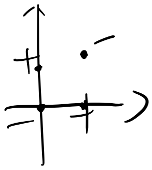
set $x_0 = 0$ $\mathrm{sign}(b) = y_0$ $\Rightarrow$ $b \geq 0$ if $y_0 = 1$ $b < 0$ o.w.

$$\mathrm{sign}(w^T x_{n+1} + b) = \mathrm{sign}\left(\sum_{i=1}^{n} k_i w^T x_i + b\right)$$

$$= \mathrm{sign}\left(k_1 w^T x_1 + k_2 w^T x_2 + \cdots + k_n w^T x_n + b\right) = y_{n+1}$$

$$= \mathrm{sign}\left(\sum_{i=1}^{n} k_i \cdot \underline{(w^T x_i + b)} + \left(1 - \sum_{i=1}^{n} k_i\right) b\right)$$

i=1,…n Set $y_i = \mathrm{sign}(k_i)$. s.t $k_i(w^Tx_i+b)>0$ (k_i v).



Set $y_0 = \mathrm{sign}(1-\sum_i k_i)$ s.t $(1-\sum_i k_i)b>0$

then y_{n+1} must be 1, we set $y_{n+1}=-1$.

$h(x_{n+1}) \neq y_{n+}$

e.g. in $\mathbb{R}^2$

$x_0=0$ $x_1=(1,0)$ $x_2=(0,1)$ $x_3=(1,1)=x_1+x_2$

$\mathrm{sign}(w^Tx_1+b)=y_1 \longrightarrow w^Tx_1+b>0$

$\mathrm{sign}(b)=y_0 \longrightarrow b<0$

$\mathrm{sign}(w^Tx_2+b)=y_2 \longrightarrow w^Tx_2+b>0.$

$\mathrm{sign}(w^Tx_1+w^Tx_2+b)=y_3$

$\mathrm{sign}(\underbrace{w^Tx_1+b}_{>0}+\underbrace{w^Tx_2+b}_{>0}\underbrace{-b}_{>0})=y_3>0 \neq$ set $y_3<0$