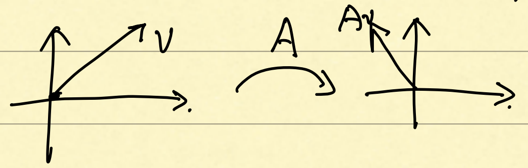
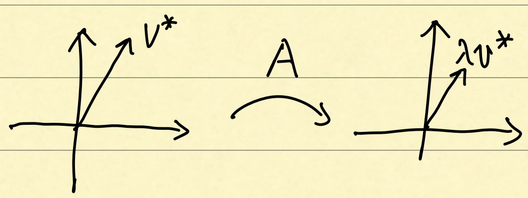
1. Eigenvalue Decomposition.

Definition 1 (Eigenvalue). $A \in \mathbb{R}^{n\times n}$, we say that $\lambda \in \mathbb{R}$ is an eigenvalue of A if there exists some non-zero vector $v \in \mathbb{R}^n \setminus \{0\}$ such that :

$$\underline{A \cdot v = \lambda v}. \qquad (*).$$



(*):



Thm 1. (Eigenvalue Decomposition). Let $\underline{A} \in \mathbb{R}^{n\times n}$ be $\underline{\text{symmetric}}$, i.e. $A^T = A$.
Then there exist orthonormal vectors $v_1, \cdots, v_n \in \mathbb{R}^n$, and real scalars $\lambda_1 \geq \cdots \geq \lambda_n$, such that if we write.

$$V \triangleq [v_1 | \cdots | v_n] \in O(n), \quad \Lambda \triangleq \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \in \mathbb{R}^{n\times n},$$

then we have :

$$A = \underline{V} \underset{\text{diag.}}{\Lambda} \underline{V^T}.$$

Notice: ① If v is unitary vector, then $\|v\|_2^2 = 1$

② If $v_1, \cdots, v_n$ is orthonormal vectors, then

$$\langle v_i, v_j \rangle = \underline{\delta_{ij}} = \begin{cases} 1, & i=j \\ 0, & \text{o.w} \end{cases}.$$

③ $V^T \cdot V = I.$

$(V^T \cdot V)_{ij} = \langle v_i, v_j \rangle = \delta_{ij}.$

④ $A = V \Lambda V^T$

$= [v_1 | \cdots | v_n] \cdot \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \cdot \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}.$

$= \sum_{i=1}^{n} \lambda_i \cdot v_i \cdot v_i^T.$

2. Singular Value Decomposition (SVD).

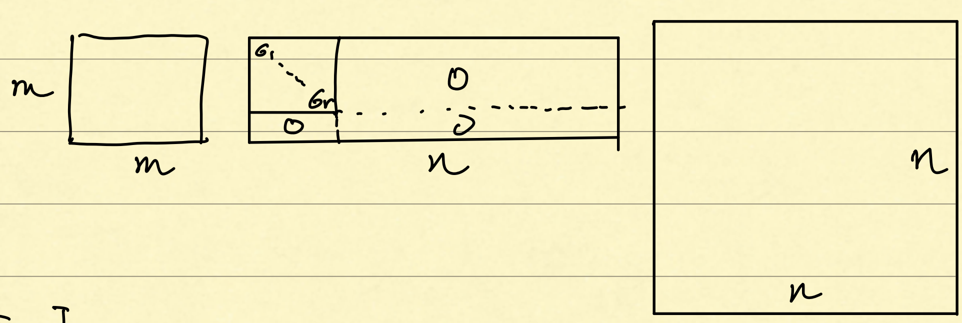
Thm2: Let $A \in \mathbb{R}^{m \times n}$, there exists orthorgnal matrix $U \in \mathbb{R}^{m \times m}$, and $V \in \mathbb{R}^{n \times n}$, such that $m \neq n.$

$$A = \underset{m\times m}{U} \cdot \underset{m\times n}{\Sigma} \cdot \underset{n\times n}{V^T};$$

where $\Sigma \triangleq \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}_{m \times n}$, $\Sigma_1 \triangleq \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}$,

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, $\rightarrow$ $r = \text{rank}(A).$

Notice: ① $A = U \cdot \Sigma \cdot V^T$



$U^T U = I_m$

$V^T V = I_n$

② $A = U \Sigma V^T$

$= [u_1 | \cdots | u_m] \cdot \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ & 0 & & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$

$= \sum_{i=1}^{r} \sigma_i \cdot u_i \cdot v_i^T .$

③ $A \cdot v_k = \left(\sum_{i=1}^{r} \sigma_i \cdot u_i \cdot v_i^T \right) \cdot v_k$, $v_i^T v_k = \delta_{ik}$.

$= \sum_{i=1}^{r} \sigma_i \cdot u_i \underbrace{\left(v_i^T \cdot v_k \right)}$ $= \begin{cases} 1 & , i=k \\ 0 & , i \neq k. \end{cases}$

$= \sigma_1 \cdot u_1 \cdot \underline{v_1^T \cdot v_k} + \cdots + \sigma_r \cdot u_r \cdot v_r^T \cdot v_k .$

$= 0 + \cdots + \sigma_k u_k + \cdots + 0$

$= \sigma_k \cdot u_k .$, $\forall k.$

$\Longleftrightarrow$ $u_k^T \cdot A = \sigma_k \cdot v_k^T$

$A^T \cdot u_k = \sigma_k \cdot v_k .$ $\forall k.$

④ $A \cdot A^T = U \cdot \Sigma \cdot \boxed{V^T \cdot V} \cdot \Sigma^T \cdot U^T .$

$= U \cdot \underline{\Sigma \cdot \Sigma^T} \cdot U^T .$

⑤ $\|A\|_F^2 \triangleq \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2$

$= tr(A^T A).$ (prove it afterwards).

$= tr(V \Sigma^T \Sigma V^T).$ ($tr(AB) = tr(BA)$).

$= tr(V^T \cdot V \cdot \Sigma^T \Sigma).$

$= tr(\Sigma^T \Sigma).$

$= \sigma_1^2 + \cdots + \sigma_r^2 .$ $\longleftarrow$

⑥ $\|A\|_{spec} \triangleq \sigma_{max} .$

Thm 3. Low-rank approximation.

$A \in \mathbb{R}^{m \times n}$, $\quad \mathrm{rank}(A) = \underline{r > k} = \mathrm{rank}(B)$.

$$\arg\min_{\mathrm{rank}(B)=k} \|A-B\|_F^2 = A_k = \sum_{i=1}^{k} \sigma_i u_i \cdot v_i^T.$$

$$\Rightarrow \min_{\substack{\mathrm{rank}(B)\\=k}} \|A-B\|_F^2 = \|A-A_k\|_F^2 = \underline{\sum_{i=k+1}^{r} \sigma_i \cdot u_i \cdot v_i^T}$$

$$= \sigma_{k+1}^2 + \cdots + \sigma_r^2.$$