# Learning From Data
# Lecture 11: Reinforcement Learning

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TBSI

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# Today's Lecture

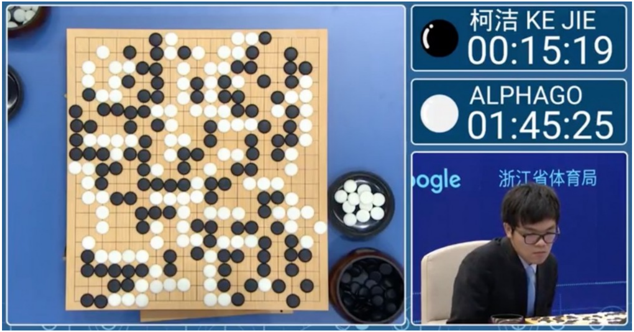
Reinforcement Learning

- What's reinforcement learning?
- Mathematical formulation: Markov Decision Process (MDP)
- Model Learning for MDP, Fitted Value Iteration
- Deep reinforcement learning (Deep Q-networks)

Final project and PA4 is released today!

# Deep Reinforcement Learning: AlphaGo

AlphaGo beat World Go Champion Kejie (2017)





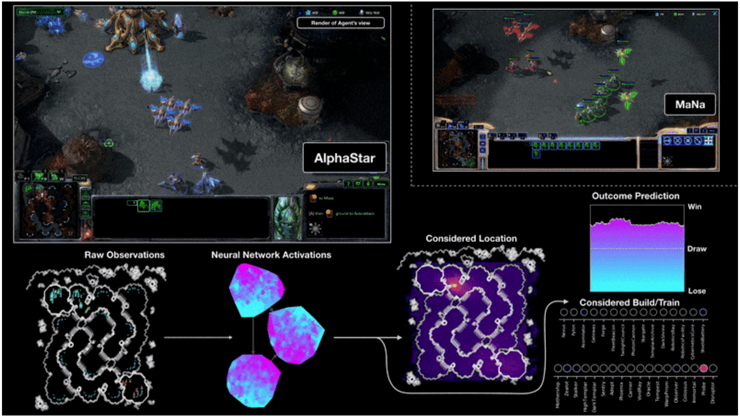
Nature paper on by AlphaGo team

# Deep Reinforcement Learning: OpenAI

OpenAI beats Dota2 world champion (2017)

# Multi-Agent Reinforcement Learning: AlphaStar

AlphaStar reached Grandmaster level in StarCraft II (2019)




https://www.nature.com/articles/s41586-019-1724-z

# Reinforcement Learning: Autonomous Car, Helicopter

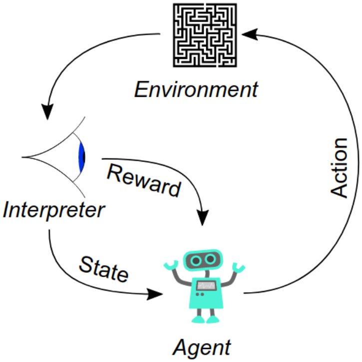
Stanley, Winner of DARPA Grand Challenge (2005)
Inverted autonomous helicopter flight (2004)

Other applications include robotic control, computational economics and etc

# What is reinforcement learning?

For sequential decision making problem, it is difficult to provide explicit supervision

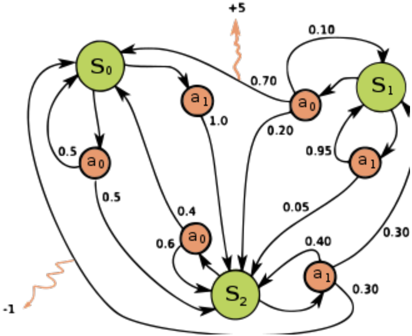
- An agent interacts with an environment which provides a "reward function" to indicate how "well" the learning agent is doing
- The agents take actions to maximize the cumulative "reward"



# Markov Decision Process

A Markov decision process $(S, A, \{P_{sa}\}, \gamma, R)$

- $S$: a set of **states** (environment)
- $A$: a set of **actions**
- $P_{sa}$: state transition probabilities.
- $R : S \times A \to \mathbb{R}$ is a **reward function**
- $\gamma \in [0, 1)$: discount factor




$S = \{S_0, S_1, S_2\}$

$A = \{a_0, a_1\}$

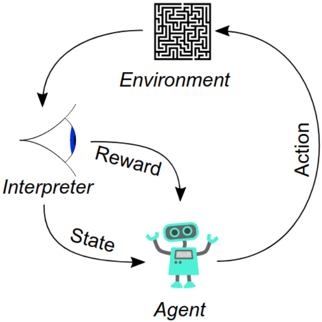
$R(s_1, a_0) = 5, R(s_2, a_1) = -1$

| | $S_0$ | $S_1$ | $S_2$ |
|---|---|---|---|
| $S_0, a_0$ | 0.5 | 0 | 0.5 |
| $S_0, a_1$ | 0 | 0 | 1 |
| $S_1, a_0$ | 0.7 | 0.1 | 0.2 |
| $S_1, a_1$ | 0 | 0.95 | 0.05 |
| $S_2, a_0$ | 0.4 | 0.6 | 0 |
| $S_2, a_1$ | 0.3 | 0.3 | 0.4 |

# Markov Decision Process: Overview

At time step $t = 0$ with initial state
$s_0 \in S$
for $t = 0$ until done:

- Agent selects action at $a_t \in A$
- Environment yields reward $r_t = R(s_t, a_t)$
- Environment samples next state $s_{t+1} \sim P_{sa}$
- Agent receives reward $r_t$ and next state $s_{t+1}$

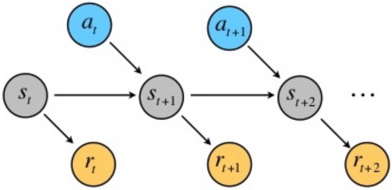



A **policy** $\pi : S \to A$ specifies what action to take in each state

Goal: find optimal policy $\pi^*$ that maximizes cumulative discounted reward

# Markov Decision Process

Consider a sequence of states $s_0, s_1, \ldots$ with actions $a_0, a_1, \ldots$,




Total payoff of a sequence:

$$R(s_0, a_0) + \gamma R(s_1, a_1) + \gamma^2 R(s_2, a_2) + \ldots$$

For simplicity, let's assume rewards only depends on state $s$, i.e.

$$R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots$$

Future reward at step $t$ is discounted by $\gamma^t$

# Policy & value functions

Goal of reinforcement learning: choose actions that maximize the expected total payoff

$$\mathbb{E}[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots]$$

A **policy** is any function $\pi \colon S \to A$.
A **value function** of policy $\pi$ is the expected payoff if we start from $s$, take actions according to $\pi$

$$V^\pi(s) = \mathbb{E}[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots | s_0 = s, \pi]$$

Given $\pi$, value function satisfies the **Bellman equation**:

$$V^\pi(s) = R(s) + \gamma \sum_{s' \in S} P_{s\pi(s)}(s') V^\pi(s')$$

*$V^\pi(s)$ can be solved as $|S|$ linear equations with $|S|$ unknowns.*

# Optimal value and policy

We define the **optimal value function**

$$V^*(s) = \max_{\pi} V^{\pi}(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s') V^*(s')$$

Let $\pi^* : S \to A$ be the policy that attains $V^*(s)$:

$$\pi^*(s) = \operatorname*{argmax}_{a \in A} \sum_{s' \in S} P_{sa}(s') V^*(s')$$

Then for every state $s$ and every policy $\pi$,

$$V^*(s) = V^{\pi^*}(s) \geq V^{\pi}(s)$$

# Solving finite-state MDP: value iteration

Assume the MDP has finite state and action space.

```
1. For each state s, initialize V(s) := 0
2. Repeat until convergence {
        Update V(s) := R(s) + max_{a∈A} γ Σ_{s'∈S} P_sa(s') V(s')
          for every state s
   }
```

Two ways to update $V(s)$:

- Synchronous update:

```
Set V_0(s) := V(s)  for all states s ∈ S
For each s ∈ S:
        V(s) := R(s) + max_{a∈A} γ Σ_{s'∈S} P_sa(s') V_0(s')
```

- Asynchronous update:

```
For each s ∈ S:
        V(s) := R(s) + max_{a∈A} γ Σ_{s'∈S} P_sa(s') V(s')
```

# Solving finite-state MDP: policy iteration

```
1. Initialize π randomly
2. Repeat until convergence {
    a. Let  V := V^π
    b. For each state s,
         π(s) := argmax_{a∈A} Σ_{s'} P_sa(s') V(s')
   }
```

Step (a) can be done by solving Bellman's equation.

# Discussion

Both value iteration and policy iteration will converge to $V^*$ and $\pi^*$

## Value iteration vs. policy iteration

- Policy iteration is more efficient and converge faster for small MDP
- Value iteration is more practical for MDP's with large state spaces

# Learning a model for finite-state MDP

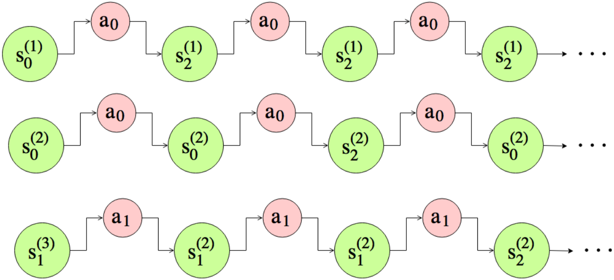
Suppose the reward function $R(s)$ and the transition probability $P_{sa}$ is not known. How to estimate them from data?

## Experience from MDP

Given policy $\pi$:

| $s$ | $\pi(s)$ |
|---|---|
| $s_0$ | $a_0$ |
| $s_1$ | $a_1$ |
| $s_2$ | $a_0$ |

Execute $\pi$ repeatedly in the environment:



# Estimate model from experience

## Estimate $P_{sa}$

Maximum likelihood estimate of state transition probability:

$$P_{sa}(s') = P(s'|s,a) = \frac{\#\{s \xrightarrow{a} s'\}}{\#\{s \xrightarrow{a} \cdot\}}$$

If $\#\{s \xrightarrow{a} \cdot\} = 0$, set $P_{sa}(s') = \frac{1}{|S|}$.

## Estimate $R(s)$

Let $R(s)^{(t)}$ be the immediate reward of state $s$ in the $t$-th trail,

$$R(s) = \mathbb{E}[R(s)^{(t)}] = \frac{1}{m}\sum_{t=1}^{m} R(s)^{(t)}$$

## Algorithm: MDP Model Learning

```
1. Initialize π randomly, V(s) := 0 for all s
2. Repeat until convergence {
   a. Execute π for m trails
   b. Update P_sa and R using the accumulated
         experience
   c. V :=ValueIteration(P_sa, R, V)
   b. Update π greedily with respect to V:
        π(s) := argmax_{a∈A} Σ_{s'} P_sa(s')V(s')
   }
```

## ValueIteration($P_{sa}, R, V_0$)

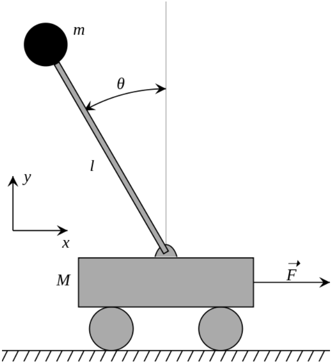
```
1. Initialize V = V_0
2. Repeat until convergence {
        Update   V(s) := R(s) + max_{a∈A} γ Σ_{s'∈S} P_sa(s')V(s')
          for every state s
   }
```

# Continuous state MDPs

An MDP may have an infinite number of states:

- A car's state : $(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$
- A helicopter's state : $(x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, \dot{\phi}, \dot{\theta}, \dot{\psi})$

## 1D Inverted Pendulum




Control goal: balance the pole on the cart

- State representation: $(x, \theta, \dot{x}, \dot{\theta})$
- Action: force $F$ on the car
- Reward: $+1$ each time the pole is upright

Due to the Curse of Dimensionality, discretization rarely works well in continuous state with more than 1-2 dimensions

# Value function approximation

How to approximate $V$ directly without resorting to discretization?

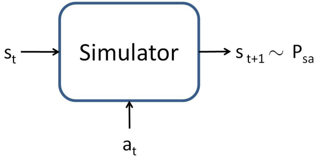
Main ideas:

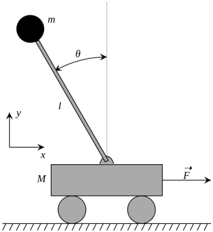
- Obtain a *model* or *simulator* for the MDP, to produce **experience tuples**: $\langle s, a, s', r \rangle$
- Sample $s^{(1)}, \ldots, s^{(m)}$ from the state space $S$, estimate their optimal expected total payoff using the model, i.e. $y^{(1)} \approx V(s^{(1)}), y^{(2)} \approx V(s^{(2)}), \ldots$
- Approximate $V$ as a function of state $s$ using supervised learning from $(s^{(1)}, y^{(1)}), (s^{(2)}, y^{(2)}), \ldots$ e.g.

$$V(s) = \theta^T \phi(s)$$

# Obtaining a simulator

A **simulator** is a black box that generates the next state $s_{t+1}$ given current state $s_t$ and action $a_t$.



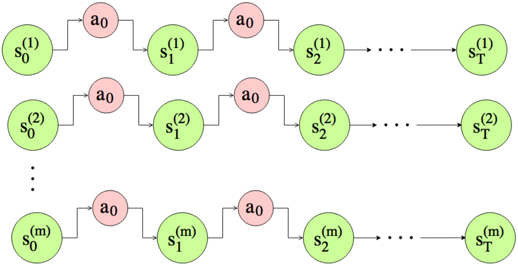

- Use physics laws. e.g. equation of motion for the inversed pendulum problem:

$$(M+m)\ddot{x} + b\dot{x} + ml\ddot{\theta}\cos(\theta) - ml\dot{\theta}^2\sin(\theta) = F$$
$$(I+ml^2)\ddot{\theta} + mgl\sin(\theta) = -ml\ddot{x}\cos(\theta)$$

- Use out-of-the-shelf simulation software
- Game simulator

# Obtaining a model from data

Execute $m$ trails in which we repeatedly take actions in an MDP, each trial for $T$ timesteps.




Learn a prediction model $s_{t+1} = h_\theta\left(\begin{bmatrix} s_t \\ a_t \end{bmatrix}\right)$ by picking

$$\theta^* = \underset{\theta}{\operatorname{argmin}} \sum_{i=1}^{m} \sum_{t=0}^{T-1} \left\| s_{t+1}^{(i)} - h_\theta\left(\begin{bmatrix} s_t^{(i)} \\ a_t^{(i)} \end{bmatrix}\right) \right\|^2$$

# Obtaining a model from data

## Popular prediction models

- Linear function: $h_\theta = As_t + Ba_t$
- Linear function with feature mapping: $h_\theta = A\phi_s(s_t) + B\phi_a(a_t)$
- Neural network

## Build a simulator using the model:

- Deterministic model: $s_{t+1} = h_\theta\left(\begin{bmatrix} s_t \\ a_t \end{bmatrix}\right)$
- Stochastic model: $s_{t+1} = h_\theta\left(\begin{bmatrix} s_t \\ a_t \end{bmatrix}\right) + \epsilon_t \ , \ \ \epsilon_t \sim \mathcal{N}(0, \Sigma)$

# Value function approximation

How to approximate $V$ directly without resorting to discretization?

Main ideas:

- Obtain a model or simulator for the MDP
- Sample $s^{(1)}, \ldots, s^{(m)}$ from the state space $S$, estimate their optimal expected total payoff using the model, i.e. $y^{(1)} \approx V(s^{(1)}), y^{(2)} \approx V(s^{(2)}), \ldots$
- Approximate $V$ as a function of state $s$ using supervised learning from $(s^{(1)}, y^{(1)}), (s^{(2)}, y^{(2)}), \ldots$ e.g.

$$V(s) = \theta^T \phi(s)$$

# Value function for continuous states

Update for finite-state value function:

$$V(s) := R(s) + \gamma \max_{a \in A} \sum_{s' \in S} P_{sa}(s') V(s')$$

Update we want for continuous-state value function:

$$\begin{aligned} V(s) &:= R(s) + \gamma \max_{a \in A} \int_{s'} P_{sa}(s') V(s') ds' \\ &= R(s) + \gamma \max_{a \in A} \mathbb{E}_{s' \sim P_{sa}} \left[ V(s') \right] \end{aligned}$$

For each sample state $s$, we compute $y^{(i)}$ to approximate $R(s) + \gamma \max_{a \in A} \mathbb{E}_{s' \sim P_{s^{(i)}a}} [V(s')]$ using finite samples from $P_{sa}$

# Value function approximation

How to approximate $V$ directly without resorting to discretization?

Main ideas:

- Obtain a model or simulator for the MDP
- Sample $s^{(1)}, \ldots, s^{(m)}$ from the state space $S$, estimate their optimal expected total payoff using the model, i.e. $y^{(1)} \approx V(s^{(1)}), y^{(2)} \approx V(s^{(2)}), \ldots$
- Approximate $V$ as a function of state $s$ using supervised learning from $(s^{(1)}, y^{(1)}), (s^{(2)}, y^{(2)}), \ldots$ e.g.

$$V(s) = \theta^T \phi(s)$$

# Fitted value iteration

## Algorithm: Fitted value iteration (Stochastic Model)

```
1.  Sample   s^(1), ..., s^(m) ∈ S
2.  Initialize θ := 0
2.  Repeat {
    a. For each sample s^(i)
        For each action a:
              Sample s'_1, ..., s'_k ~ P_{s^(i),a} using a model
              Compute  Q(a) = (1/k) Σ_{j=1}^k R(s^(i)) + γV(s'_j)
                         ↑ estimates R(s^(i)) + γ E_{s'~P_{s'a}}[V(s')]
                         where V(s) := θ^T φ(s)

        y^(i) = max_a Q(a)
         ↑ estimates R(s^(i)) + γ max_a E_{s'~P_{s'a}}[V(s')]
    b. Update θ using supervised learning:
         θ := argmin_θ (1/2) Σ_{i=1}^m (θ^T φ(s^(i)) − y^(i))^2
    }
```

Where:

$Q(a) = \frac{1}{k}\sum_{j=1}^{k} R(s^{(i)}) + \gamma V(s'_j)$ estimates $R(s^{(i)}) + \gamma \mathbb{E}_{s' \sim P_{s'a}}[V(s')]$ where $V(s) := \theta^T \phi(s)$

$y^{(i)} = \max_a Q(a)$ estimates $R(s^{(i)}) + \gamma \max_a \mathbb{E}_{s' \sim P_{s'a}}[V(s')]$

$\theta := \text{argmin}_\theta \frac{1}{2}\sum_{i=1}^{m}(\theta^T \phi(s^{(i)}) - y^{(i)})^2$

If the model is deterministic, set $k = 1$

# Computing the optimal policy

After obtaining the value function approximation $V$, the corresponding policy is

$$\pi(s) = \underset{a}{\operatorname{argmax}} \, \mathbb{E}_{s' \sim P_{sa}}[V(s')])$$

Estimate the optimal policy from experience:

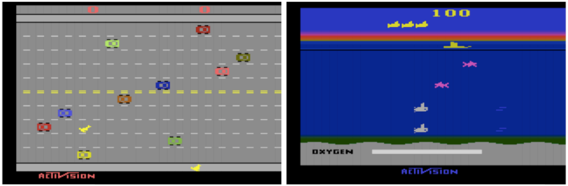
For each action $a$ :
    1. Sample $s'_1, \ldots, s'_k \sim P_{s,a}$ using a model
    2. Compute $Q(a) = \frac{1}{k} \sum_{j=1}^{k} R(s) + \gamma V(s'_j)$
$\pi(s) = \operatorname{argmax}_a Q(a)$

Instead of linear regression, other learning algorithms can be used to estimate $V(s)$.

# Two Outstanding Success Stories

## Atari AI [Minh et al. 2015]

- Plays a variety of Atari 2600 video games at superhuman level
- Trained directly from image pixels, based on a single reward signal




## AlphaGo [Silver et al. 2016]

- A hybrid deep RL system
- Trained using supervised and reinforcement learning, in combination with a traditional tree-search algorithm.

# Deep Reinforcement Learning

Main difference from classic RL:

- Use deep network to represent value function
- Optimize value function end-to-end
- Use stochastic gradient descent

# Q-Value Function

Given policy $\pi$ which produce sample sequence $(s_0, a_0, r_0), (s_1, a_1, r_1), \ldots$

- Value function of $\pi$ :

$$V^\pi(s) = \mathbb{E}\left[\sum_{t\geq 0} \gamma^t r_t \middle| s_0 = s, \pi\right]$$

- The **Q-value function** $Q^\pi(s, a)$ is the expected payoff if we take $a$ at state $s$ and follow $\pi$

$$Q^\pi(s, a) = \mathbb{E}\left[\sum_{t\geq 0} \gamma^t r_t \middle| s_0 = s, a_0 = a, \pi\right]$$

- The optimal Q-value function is:

$$Q^*(s, a) = \max_\pi Q^\pi(s, a) = \max_\pi \mathbb{E}\left[\sum_{t\geq 0} \gamma^t r_t \middle| s_0 = s, a_0 = a, \pi\right]$$

# Q-Learning

Bellman's equation for Q-Value function:

$$Q^*(s, a) = \mathbb{E}_{s' \sim \mathcal{E}}[r + \gamma \max_{a'} Q^*(s', a') | s, a]$$

Value iteration is not practical when the search space is large.

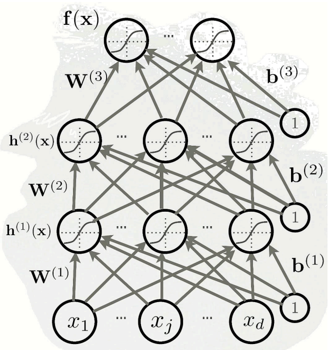
e.g. In an Atari game, each frame is an 128-color $210 \times 160$ image, then $|S| = 128^{210 \times 160}$

- Uses a function approximation:

$$Q(s, a; \theta) \approx Q^*(s, a)$$

- In deep Q-learning, $Q(s, a; \theta)$ is a neural network

# Neural Network Review




Training goal:
$\min_\theta \sum_{i=1}^m L(f(x^{(i)}; \theta), y^{(i)})$

## Forward propagation

Initialize $h^{(0)}(x) = x$
For each layer $l = 1 \dots d$:

- $a^{(l)}(x) = W^{(l)} h^{(l-1)}(x) + b^{(l)}$
- $h^{(l)}(x) = g(a^{(l)}(x))$

Evaluate loss function $L(h^{(d)}(x), y)$

## Backward propagation

Compute gradient $\frac{dL}{dh^{(d)}}$
For each layer $l = d \dots 1$:

- Update gradient for parameters in layer $l$

# Q-Networks

Training goal: find $Q(s, a; \theta)$ that fits Bellman's equation:
$Q^*(s, a) = \mathbb{E}_{s' \sim \mathcal{E}}[r + \gamma \max_{a'} Q^*(s', a')|s, a]$

## Forward Pass

Loss function:

$$L_i(\theta_i) = \mathbb{E}_{s,a}[(y_i - Q(s, a; \theta_i)^2]$$

where $y_i = \mathbb{E}_{s' \sim \mathcal{E}}[r + \gamma \max_{a'} Q(s', a'; \theta_{i-1})|s, a]$

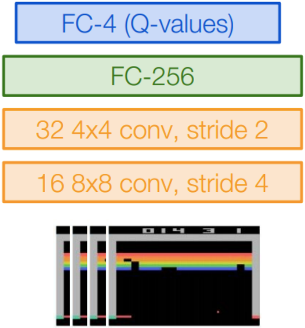
## Backward Pass

Update parameter $\theta$ by computing gradient

$$\nabla_{\theta_i} L_i(\theta_i) = \mathbb{E}_{s,a,s' \sim \mathcal{E}} \left[ \left( r + \gamma \max_{a'} Q(s', a'; \theta_{i-1}) - Q(s, a; \theta_i) \right) \nabla_\theta Q(s, a; \theta_i) \right]$$

# Deep Q-Network Architecture

- Input: 4 consecutive frames
- Preprocessing: convert to grayscale, down-sampling, cropping. Final dimension $84 \times 84 \times 4$
- Output: Q-value functions for 4 actions $Q(s, a_1)$, $Q(s, a_2)$, $Q(s, a_3)$, $Q(s, a_4)$



# Experience Replay

Challenge of standard deep Q-learning: correlated input

- invalidate the i.i.d. assumption on training samples
- current policy may restrict action samples we experience in the environment

Experience replay

- Store past transitions $(s_t, a_t, r_t, s_{t+1})$ within a sliding window in the **replay memory** $D$.
- Train Q-Network using random mini-batch sampled from $D$ to reduce sample correlation
- Also reduces total running time by reusing samples

# The Algorithm

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**Algorithm 1** Deep Q-learning with Experience Replay

---

Initialize replay memory $\mathcal{D}$ to capacity $N$
Initialize action-value function $Q$ with random weights
**for** episode $= 1, M$ **do**
  Initialise sequence $s_1 = \{x_1\}$ and preprocessed sequenced $\phi_1 = \phi(s_1)$
  **for** $t = 1, T$ **do**
    With probability $\epsilon$ select a random action $a_t$
    otherwise select $a_t = \max_a Q^*(\phi(s_t), a; \theta)$
    Execute action $a_t$ in emulator and observe reward $r_t$ and image $x_{t+1}$
    Set $s_{t+1} = s_t, a_t, x_{t+1}$ and preprocess $\phi_{t+1} = \phi(s_{t+1})$
    Store transition $(\phi_t, a_t, r_t, \phi_{t+1})$ in $\mathcal{D}$
    Sample random minibatch of transitions $(\phi_j, a_j, r_j, \phi_{j+1})$ from $\mathcal{D}$
    Set $y_j = \begin{cases} r_j & \text{for terminal } \phi_{j+1} \\ r_j + \gamma \max_{a'} Q(\phi_{j+1}, a'; \theta) & \text{for non-terminal } \phi_{j+1} \end{cases}$
    Perform a gradient descent step on $(y_j - Q(\phi_j, a_j; \theta))^2$ according to equation 3
  **end for**
**end for**

---

Parameter $\epsilon$ controls the exploration vs. optimization trade-off

# Reinforcement Learning Demo

See Demo.