Tsinghua-Berkeley Shenzhen Institute
LEARNING FROM DATA
Fall 2020

Homework 0

Issued: Saturday 12th September, 2020 **Due:** Thursday 17th September, 2020

POLICIES

- **Acknowledgments:** We expect you to make an honest effort to solve the problems individually. As we sometimes reuse problem set questions from previous years, covered by papers and web pages, we expect the students **NOT** to copy, refer to, or look at the solutions in preparing their answers (relating to an unauthorized material is considered a violation of the honor principle). Similarly, we expect to not to google directly for answers (though you are free to google for knowledge about the topic). If you do happen to use other material, it must be acknowledged here, with a citation on the submitted solution.

- **Required homework submission format:** You can submit homework either as one single PDF document or as handwritten papers. Written homework needs to be provided during the class in the due date, and PDF document needs to be submitted through Tsinghua's Web Learning (`http://learn.tsinghua.edu.cn/`) before the end of due date.

  It is encouraged you LaTeX all your work, and we would provide a LaTeX template for your homework.

- **Collaborators:** In a separate section (before your answers), list the names of all people you collaborated with and for which question(s). If you did the HW entirely on your own, **PLEASE STATE THIS**. Each student must understand, write, and hand in answers of their own.

Tips: It is not a formal homework and will not be graded. The primary goal is to help you remember those basic mathematics you have learnt before.

Calculus & Linear Algebra

0.1. (Inner product) If $\boldsymbol{x} \in \mathbb{R}^n$ is orthogonal to $\boldsymbol{y} \in \mathbb{R}^n$, please show that

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2$$

0.2. (Orthogonal) Please show that $\|\mathbf{Q}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$.

0.3. (Trace) For any matrices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$, please show that

   (a) $\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$

(b) $\text{trace}(\mathbf{ABC}) = \text{trace}(\mathbf{CAB}) = \text{trace}(\mathbf{BCA})$

(c) $\nabla_{\mathbf{A}} \text{tr}(\mathbf{AB}) = \mathbf{B}^{\mathrm{T}}$.

0.4. (Eigenthings) Let $\boldsymbol{x}$ be an eigenvector of a matrix $\mathbf{A}$ with corresponding eigenvalue λ, then

(a) Show that for any $\gamma \in \mathbb{R}$, the $\boldsymbol{x}$ is an eigenvector of $\mathbf{A} + \gamma I$ with eigenvalue $\lambda + \gamma$.

(b) If $\mathbf{A}$ is invertible, then $\boldsymbol{x}$ is an eigenvector of $\mathbf{A}^{-1}$ with eigenvalue λ^{-1}.

(c) $\mathbf{A}^k \boldsymbol{x} = \lambda^k \boldsymbol{x}$ for any $k \in \mathbb{Z}$ ($\mathbf{A}^0 = I$ by definition)

0.5. (Chain rule) $x \in \mathbb{R}$ is a scalar, we have

$$y = ax + b$$
$$z = \frac{1}{1 + e^{-y}}$$

Please give the $\frac{\partial z}{\partial x}$.

0.6. (Matrix derivative) $\boldsymbol{x}, \boldsymbol{w} \in \mathbb{R}^n$, and $\mathbf{A} \in \mathbb{R}^{n \times n}$. We have $f : \mathbb{R}^n \to \mathbb{R}$ as

$$f(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} \mathbf{A} \boldsymbol{x} + \boldsymbol{w}^{\mathrm{T}} \boldsymbol{x}$$

Please give the $\nabla_{\boldsymbol{x}} f(\boldsymbol{x})$.

Probability Theory Part

0.7. (Conditional Probability) Explain that $\mathbb{E}\left[\mathbb{E}\left[X|Y\right]\right] = \mathbb{E}[X]$

0.8. (Bayes) A city has a 50% chance to rain everyday and the weather report has a 90% chance to correctly forecast.
You will take an umbrella when the report says it will rain and you have a 50% chance to take an umbrella when the report says it will not rain.
Compute

(a) the probability of raining when you don't take an umbralla;

(b) the probability of not rainning when you take an umbrella.

0.9. (Joint Distribution) Random Variables X and Y have a joint distribution with joint probability density function

$$f(x, y) = \begin{cases} Ce^{-(2x+y)} & x > 0, y > 0 \\ 0 & ow. \end{cases}$$

Please find C by

$$\int_0^\infty \int_0^\infty f(x, y) \mathrm{d}x \mathrm{d}y = 1$$

0.10. (Covariance) Now we have a joint pdf

$$f(x,y) = \begin{cases} 4xy & 0 < x < 1, 0 < y < 1 \\ 0 & ow. \end{cases}$$

Please show that the covariance of X and Y is 0.

0.11. (Uncorrelated and independent RVs) We have a uniform distribution of X and Y on a disk. The pdf is

$$f(x,y) = \frac{1}{\pi} \qquad x^2 + y^2 \leq 1$$

Please show that X and Y are uncorrelated but not independent.

0.12. (Guassian Distribution) There is a famous integral here

$$\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}$$

It is called Guassian Integral. Based on it, please find some results of the Normal Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad -\infty < x < \infty$$

(a) Prove it is a pdf ($\sigma > 0$)

(b) Compute the expectation and variance